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Polynomial Inequalities on Measurable Sets and Their Applications II. Weighted Measures

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We study Remez-type inequalities for univariate and multivariate polynomials on bounded sets in weighted spaces and discuss their applications to Nikolskii-type inequalities and local approximation.

Key Words: Remez- and Nikolskii-type inequalities; weighted measures; polynomial approximation.

1. INTRODUCTION

The paper is devoted to Remez-type inequalities on bounded sets in weighted spaces and their applications in approximation theory.

Weighted inequalities for polynomials have received much attention since the 1950s. It was Dzyadyk [15, 16, 47] who showed that such inequalities for the Timan weight $(\sqrt{1-x^2}+n^{-1})^\alpha$ played an important role in inverse theorems of approximation theory. In the 1950s–1970s Lebed, Potapov, Daugavet, Rafalson, Konyagin, and others developed the Markov–Nikolskii inequalities for polynomials in one and several variables in the L_p-spaces, equipped with some special weighted measures. For all this, see [11–14, 33]. A Nikolskii-type inequality for a general weight and multivariate polynomials was established by the author [23]. For the past 15 years considerable attention has been devoted to the Bernstein–Markov and Nikolskii inequalities for the exponential weights on an interval or the real axis (see [35, 36, 44] for discussions and references).

In the 1970s the author [21, 22] developed a new approach to Nikolskii-type inequalities in rearrangement-invariant spaces based on Remez-type

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inequalities. We discussed these estimates, their generalizations and applications to some problems of Analysis in part I of this paper [24]. However, the weighted analogues of these results are unknown.

In this part we extend the Remez inequalities to a general weighted measure on a bounded set and discuss their applications to Nikolskii-type inequalities and local approximation.

The paper is organized as follows: Section 2 contains Remez-type inequalities for algebraic and trigonometric polynomials on bounded sets equipped with a weighted measure μ. In particular, we prove weighted estimates for multivariate polynomials (Theorem 2.1(a), Corollaries 2.1 and 2.2) and show that the corresponding constant in Theorem 2.1(a) is sharp on the class of all convex bodies (Theorem 2.2(b)). We also establish the homogeneous inequality for algebraic polynomials P of degree n and a measurable subset E of a convex body V of the form $\|P\|_{C(V)} \leqslant C(\mu V/\mu E)^n \|P\|_{C(E)}$, provided μ satisfies a special condition (Theorem 2.2(a)). It is shown that for certain classes of weights, the condition is necessary (Theorem 2.2(b)). The Remez-type inequalities for trigonometric polynomials (Theorem 2.3) and estimates of μ-rearrangements (Corollaries 2.3, 2.4, and 2.5) are also presented.

In Section 3 we obtain the Nikolskii- and Schur-type inequalities in weighted rearrangement-invariant spaces for algebraic and trigonometric polynomials (Theorems 3.1 and 3.2, Corollaries 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6).

Section 4 is devoted to some applications of the homogeneous inequality of the Nikolskii type (Corollary 3.3) in approximation theory. We remark that the problems of obtaining the homogeneous inequalities for polynomials and their applications to local approximation have been communicated to the author by Yu. A. Brudnyi in the early 1970s.

In Section 5 we apply the general estimates, established in Sections 2 and 3, to the following particular weights: a generalized Jacobi weight $\prod_{i=1}^{k} |x-x_i|^{\alpha_i}$, a Jacobi weight $(1-x)^{\alpha}(1-x)^{\beta}$, an exponential weight $x^{\lambda}\exp(-x^{-\alpha})$, and a generalized Gegenbauer–Timan weight $(\rho(x,\Gamma))^{\alpha}(\rho(x,\Gamma)+n^{-2m})^{\beta}$, where Γ is an s-dimensional surface in $\mathbf{R}^m$.

Notation and definitions. We use the following notation.

Let $\mathbf{R}^m$ be the m-dimensional Euclidean space; $B_r(y) := \{x \in \mathbf{R}^m : |x-y| \leqslant r\}$ an m-dimensional ball in $\mathbf{R}^m$ of radius r centered at y; $Q_r(y) = Q(y) := \{x \in \mathbf{R}^m : |x_i - y_i)| \leqslant r/2, 1 \leqslant i \leqslant m\}$ a cube in $\mathbf{R}^m$ with the edge length r; $\rho(x, E) := \inf_{y \in E} |x-y|$ the distance from $x \in \mathbf{R}^m$ to a set $E \subseteq \mathbf{R}^m$; $\rho(Q, E) := \inf_{x \in Q} \rho(x, E)$, the distance between sets Q and E in $\mathbf{R}^m$; $|E|_k$ k-dimensional Lebesgue measure of an L-measurable set $E \subset \mathbf{R}^m$, $1 \leqslant k \leqslant m$; χ_E the characteristic function of $E \subseteq \mathbf{R}^m$.

Let $\mathscr{P}_{n,m}$ be the class of all algebraic polynomials in m variables with real coefficients of degree n; $\mathscr{T}_n$ the class of all trigonometric polynomials of a single variable of degree n with real coefficients. We also make use of the Chebyshev polynomial

$$T_n(\tau) := (1/2)((\tau + \sqrt{\tau^2 - 1})^n + (\tau - \sqrt{\tau^2 - 1})^n)$$

of degree n.

Throughout the paper W denotes an integrable weight defined on a set $\Omega \subseteq \mathbf{R}^m$ with the property: $|\{x \in \Omega : W(x) = 0\}|_m = 0$. The weighted measure μ of a Lebesgue-measurable (L-measurable) set $E \subseteq \Omega$ is defined by $\mu E = \int_E W(x)\, dx$.

Further, let $C(\Omega)$ be the real space of all real valued continuous functions f on $\Omega \subseteq \mathbf{R}^m$ with the norm $\|f\|_{C(\Omega)} := \sup_{x \in \Omega} |f(x)|$, and let $L_{p,W}(\Omega)$, $0 < p \leqslant \infty$, be the space of all L-measurable functions f on $\Omega \subseteq \mathbf{R}^m$ such that $\|f\|_{L_{p,W}(\Omega)} := (\int_\Omega |f(x)|^p\, W(x)\, dx)^{1/p} < \infty$ if $0 < p < \infty$, and $L_{\infty, W}(\Omega) := C(\Omega)$. Set $L_p(\Omega) := L_{p,1}(\Omega)$, $0 < p \leqslant \infty$.

Throughout the paper $C, C_1, C_2, \ldots$ denote positive constants independent of variables n, x, t, u, τ, set E, functions and polynomials f, P, T, and occasionally also independent of sets V and Ω. The same symbol does not necessarily denote the same constant in different occurrences.

Next, we define rearrangements of functions and rearrangement-invariant spaces (see [9, 15, 30]). We consider L-measurable functions f defined on the set $\Omega \subseteq \mathbf{R}^m$, equipped with the weighted measure μ.

For each L-measurable f on the bounded set $\Omega \subset \mathbf{R}^m$ we define its *increasing μ-rearrangement* $f^*_{\mu,\Omega}$: $[0, \mu\Omega] \to [0, \infty]$ by $f^*_{\mu,\Omega}(t) := f^*_\mu(t) := \sup\{\tau \geqslant 0 : E_\tau \leqslant t\}$, where $E_\tau := \mu\{x \in \Omega : |f(x)| \leqslant \tau\}|$. If μ is the Lebesgue measure, then the corresponding rearrangement is denoted by $f^*_\Omega = f^*$. Similarly, for each L-measurable f on $\Omega \subseteq \mathbf{R}^m$ we define its *decreasing μ-rearrangement* by $f_{*\,\Omega}(t) := \inf\{\tau \geqslant 0 : I_\tau \leqslant t\}$, where $I_\tau := \mu\{x \in \Omega : |f(x)| > \tau\}$.

Let $\varphi = \varphi(\cdot, \Omega, W)$: $[0, \mu\Omega] \to [0, |\Omega|_m]$ be the inverse of the function $\int_{|\Omega|_m - y}^{|\Omega|_m} W^*_\Omega(\tau)\, d\tau$, $y \in [0, |\Omega|_m]$.

We say that a linear real space $F(\Omega)$ of L-measurable functions, defined on $\Omega \subseteq \mathbf{R}^m$, is a *weighted rearrangement-invariant* (*WRI*) *space* if there is a nonnegative functional $\|\cdot\|_{F(\Omega)}$ on $F(\Omega)$ with the properties:

(i) $\|f\|_{F(\Omega)} = 0$ if and only if $f = 0$,

(ii) $\|cf\|_{F(\Omega)} = |c|\,\|f\|_{F(\Omega)}$ for a scalar c,

(iii) if $g \in F(\Omega)$ and $f^*_\mu(t) \leqslant g^*_\mu(t)$ for all $t \in [0, \mu\Omega)$, then $f \in F(\Omega)$ and $\|f\|_{F(\Omega)} \leqslant \|g\|_{F(\Omega)}$.

The fundamental function of $F(\Omega)$ is defined by $\psi_F(t) := \|\chi_E\|_{F(\Omega)}$, where $E \subseteq \Omega$ and $\mu E = t$, $0 \leqslant t \leqslant \mu\Omega$.

Let Ω be a bounded set and $\omega: \Omega \to [0, \mu\Omega]$ a μ-measure preserving transformation which is one-to-one and onto. Then each WRI space $F(\Omega)$ generates the WRI space $\tilde{F}(0, \mu\Omega) := \{h = f(\omega\,\cdot\,) : f \in F(\Omega)\}$ with $\|h\|_{\tilde{F}(0, \mu\Omega)} = \|h \circ \omega^{-1}\|_{F(\Omega)}$ and $\psi_{\tilde{F}} = \psi_F$. It is known that $f^*_\mu \in \tilde{F}(0, \mu\Omega)$ for each $f \in F(\Omega)$ and the following generalization of the Steffensen inequality [2] holds

$$\|f\|_{F(E)} := \|f\chi_E\|_{F(\Omega)} \geqslant \|f^*_\mu \chi_E\|_{\tilde{F}(0, \mu\Omega)} := \|f^*_\mu\|_{\tilde{F}(0, \mu E)}. \tag{1.1}$$

If $F(\Omega)$ is a normed WRI (NWRI) space, that is $\|\cdot\|_{F(\Omega)}$ satisfies the triangle inequality, then [9, 30]

$$(1/2)\,\hat{\psi}_F(t) \leqslant \psi_F(t) \leqslant \hat{\psi}_F(t) \tag{1.2}$$

for all $t \geqslant 0$, where $\hat{\psi}_F$ is the least concave majorant of ψ_F.

It is easy to see that $L_{p,\,W}(\Omega)$, $0 < p < 1$, are WRI spaces, while $C(\Omega)$, $L_{p,\,W}(\Omega)$, $1 \leqslant p < \infty$, and the weighted Orlicz, Lorentz, and Marcinkiewicz spaces are NWRI.

2. REMEZ-TYPE INEQUALITIES FOR WEIGHTED MEASURES

2.1. *Algebraic Polynomials*

A convex hull of a vertex $x_0 \in \mathbf{R}^m$ and a convex $(m-1)$-dimensional body B_{m-1} (the base), $x_0 \notin B_{m-1}$, is called the bounded convex cone (BCC). The set of all BCC in $\mathbf{R}^m$ is denoted by $\mathscr{K}$.

Following is a weighted Remez-type inequality for algebraic polynomials in m variables.

THEOREM 2.1. (a) *For a polynomial $P \in \mathscr{P}_{n,\,m}$, a convex body $V \subset \mathbf{R}^m$ and an L-measurable set $E \subseteq V$, $|E|_m > 0$,*

$$\|P\|_{C(V)} \leqslant T_n\left(\frac{1 + \beta_m(\varphi(\mu E)/\varphi(\mu V))}{1 - \beta_m(\varphi(\mu E)/\varphi(\mu V))}\right) \|P\|_{C(E)}, \tag{2.1}$$

where

$$\beta_m(t) = (1-t)^{1/m}. \tag{2.2}$$

(b) *Equality in* (2.1) *holds for all $\mu E \in (0, \mu V)$, if and only if $V \in \mathscr{K}$, $W(x) = g((c, x - x_0))$, E is a layer adjacent to the base of the BCC, and $P(x) = A T_n(2(h-t)/d - 1)$, $A \in \mathbf{R}^1$. Here V is the BCC with a vertex $x_0 \in \mathbf{R}^m$,* *and a base lying in the plane* $(c, x-x_0)-h=0$, $c=(c_1, ..., c_m)\in \mathbf{R}^m$, $|c|=1$, $h>0$; *g a nondecreasing function of a single variable on* $[0, h]$; *d width of the layer E*; *and t a coordinate on the segment* $\{x\in \mathbf{R}^m : x-x_0=tc, 0\leqslant t\leqslant h\}$.

It was Remez [42] who in 1936 initiated the study of polynomial inequalities on measurable sets by proving Theorem 2.1 for $m=1$, and $W(x)=1$. Various generalizations of Remez's inequalities and applications in analysis have received much attention since the 1970's. For all this, see [3, 6, 7, 19, 22, 24, 34, 37]. In particular, Brudnyi and the author [6, 7] established m-dimensional inequality (2.1) for the nonweighted case (that is $\varphi(\mu E)=|E|_m$). Theorem 2(b) for $m>1$ and $W(x)=1$, was obtained by the author [7].

Proof of Theorem 2.1. We first prove statement (a). We shall derive inequality (2.1) from the nonweighted version of (2.1) (see [6, 7]),

$$\|P\|_{C(V)}\leqslant T_n\left(\frac{1+\beta_m(|E|_m/|V|_m)}{1-\beta_m(|E|_m/|V|_m)}\right)\|P\|_{C(E)}, \tag{2.3}$$

where β_m is defined by (2.2). Indeed, φ is strictly increasing on $[0, |V|_m]$. Hence

$$\frac{|E|_m}{|V|_m}=\frac{\varphi(\int_{|V|_m-|E|_m}^{|V|_m} W^*(\tau)\,d\tau)}{\varphi(\mu V)}\geqslant\frac{\varphi(\int_E W(\tau)\,d\tau)}{\varphi(\mu V)}=\frac{\varphi(\mu E)}{\varphi(\mu V)}. \tag{2.4}$$

Thus (2.3) and (2.4) yield (2.1).

The proof of statement (b) consists of 5 steps.

Step 1. We first prove sufficiency of statement (b). Let W, V, E, and P satisfy the conditions of statement (b). Then it is easy to verify that

$$\int_{|V|_m-|E|_m}^{|V|_m} W^*(\tau)\,d\tau=\sup_{\Omega\subset V,\ |\Omega|_m=|E|_m}\int_\Omega W(x)\,dx=\int_E W(x)\,dx=\mu E.$$

Hence $|E|_m/|V|_m=\varphi(\mu E)/\varphi(\mu V)$. Finally,

$$\begin{aligned}|P(x_0)|&=T_n\left(\frac{1+\beta_m(|E|_m/|V|_m)}{1-\beta_m(|E|_m/|V|_m)}\right)\|P\|_{C(E)}\\&=T_n\left(\frac{1+\beta_m(\varphi(\mu E)/\varphi(\mu V))}{1-\beta_m(\varphi(\mu E)/\varphi(\mu V))}\right)\|P\|_{C(E)}.\end{aligned}$$

Step 2. Next, we shall show that necessity of statement (b) can be reduced to the nonweighted case.

Let V be a convex body in $\mathbf{R}^m$, and W a weight such that for some $\mu \in (0, \mu V)$ there exist an L-measurable set $E \subset V$, $\mu E = \mu$, a point $x_0 \in V$, and a polynomial $P \in \mathscr{P}_{n,m}$, satisfying the equality

$$|P(x_0)| = T_n\left(\frac{1+\beta_m(\varphi(\mu)/\varphi(\mu V))}{1-\beta_m(\varphi(\mu)/\varphi(\mu V))}\right) \|P\|_{C(E)}. \tag{2.5}$$

Then taking account of monotonicity of $T_n(t)$ for $t > 1$, we obtain from (2.3), (2.4), and (2.5) that

$$|E|_m = \varphi(\mu), \tag{2.6}$$

$$|P(x_0)| = T_n\left(\frac{1+\beta_m(|E|_m/|V|_m)}{1-\beta_m(|E|_m/|V|_m)}\right) \|P\|_{C(E)}. \tag{2.7}$$

Step 3. Let $r = \rho(\theta) = \rho(\theta_1, ..., \theta_m)$ be the equation for the boundary of V in a spherical coordinate system with center x_0. Let us examine the set E' which in the coordinates (r, θ) is defined by $\beta_m(|E|_m/|V|_m)\,\rho(\theta) \leqslant r \leqslant \rho(\theta)$. Note first that the restriction of P to a ray l emanating from x_0 belongs to $\mathscr{P}_{m,1}$. Hence (2.7) and Remez's theorem [3, p. 228] imply

$$\mathit{ess}\inf_l |V \cap l|_1/|E \cap l|_1 \geqslant (1-\beta_m(|E|_m/|V|_m))^{-1}. \tag{2.8}$$

On the other hand, the following inequality holds [6, Lemma 3]:

$$\mathit{ess}\inf_l |V \cap l|_1/|E \cap l|_1 \leqslant (1-\beta_m(|E|_m/|V|_m))^{-1}. \tag{2.9}$$

Then (2.8) and (2.9) imply

$$\mathit{ess}\inf_l |V \cap l|_1/|E \cap l|_1 = \mathit{ess}\inf_l |V \cap l|_1/|E' \cap l|_1 = (1-\beta_m(|E|_m/|V|_m))^{-1}. \tag{2.10}$$

Taking account of the relation $|E|_m = |E'|_m$, we obtain from (2.10) that

$$|E \mathbin{\Delta} E'|_m = 0. \tag{2.11}$$

Step 4. Next, we shall show that $V \in \mathscr{K}$. Without loss of generality we may assume that $x_0 \in V$ coincides with the origin. Then taking into account the extremal properties of the Chebyshev polynomial [3, p. 235], we obtain from (2.7) and (2.11) that P coincides with the following function

$$G(x) = T_n\left(\frac{2M_V(x)}{1-\beta_m(|E|_m/|V|_m)} - \frac{1+\beta_m(|E|_m/|V|_m)}{1-\beta_m(|E|_m/|V|_m)}\right)$$

for all $x \in V$, where M_V is the Minkovsky functional of V. Remind that for x that belongs to a ray l traced from the origin, $M_V(x) := |x|/|V \cap l|_1$ (the equivalent definition is given in [43]).

Next we note that G coincides with a polynomial from $\mathscr{P}_{n,m}$ for all $x \in V$ if and only if $M_V(x) = (b, x)$ for some $b \in \mathbf{R}^m$. Indeed, if $G|_V \in \mathscr{P}_{n,m}$, then homogeneity of M_V implies that for every $\tau \in (0, 1)$ and any $x \in V$,

$$G(\tau x) = \sum_{k=0}^{n} a_k \tau^k (M_V(x))^k = \sum_{k=0}^{n} \alpha_k \tau^k P_k(x), \tag{2.12}$$

where a_k, α_k, $0 \leqslant k \leqslant n$, are real numbers,

$$a_1 = \frac{2}{1 - \beta_m(|E|_m/|V|_m)} T'_n \left(\frac{1 + \beta_m(|E|_m/|V|_m)}{1 - \beta_m(|E|_m/|V|_m)} \right) > 0,$$

and P_k are homogeneous polynomials of degree k, $0 \leqslant k \leqslant n$. In particular, (2.12) shows that $M_V(x) = CP_1(x)$, $x \in V$. Hence $M_V(x) = (b, x)$ for some $b \in \mathbf{R}^m$.

Thus $V \in \mathscr{K}$ since $V \in \mathscr{K}$ with the vertex at the origin if and only if M_V is a homogeneous polynomial of degree 1.

Step 5. Thus we proved that if V, W, E, and P satisfy (2.5) for some $\mu = \mu E \in (0, |V|_m)$, then $V \in \mathscr{K}$. Moreover, (2.11) implies that E coincides with a layer adjacent to the base of the BCC and $P(x) = AT_n(2(h-t)/d - 1)$.

It remains to show that if (2.5) holds for all μE, $0 < \mu E < \mu V$, then $W(x) = g((c, x - x_0))$, where g is a nondecreasing function of a single variable on $[0, h]$. Indeed, it follows from (2.6) that

$$\sup_{\Omega \subset V,\, |\Omega|_m = |E|_m} \int_\Omega W(x)\, dx = \int_{|V|_m - |E|_m}^{|V|_m} W^*(\tau)\, d\tau = \int_E W(x)\, dx$$

for every layer E adjacent to the base of V. This yields $W(x) = W^*(|E|_m)$ for almost all x from the set $V \cap \{x \in \mathbf{R}^m : (c, x - x_0) = h - d\}$, where h is height of the BCC V and d width of E, $0 \leqslant d \leqslant h$. Hence $W(x) = g(h-d)$, and $g(h - d_1) \geqslant g(h - d_2)$ for $d_1 \leqslant d_2$. It completes the proof of statement (b). ∎

Remark 2.1. In the proof of Theorem 2.1(b) we used some ideas from [7, Theorem 2].

Following is a special case of Theorem 2.1 for $m = 1$.

COROLLARY 2.1. *For $P \in \mathscr{P}_{n,1}$ and $E \subseteq V = [a, b]$, $|E|_1 > 0$,*

$$\|P\|_{C(a,b)} \leqslant T_n\left(\frac{2\varphi(\mu V)}{\varphi(\mu E)} - 1\right) \|P\|_{C(E)}. \tag{2.13}$$

Equality in (2.13) *for $\mu E = \mu$, $0 < \mu < \mu V$ holds if and only if $E = [a, a + \varphi(\mu)]$, $P(x) = AT_n(\frac{2(x-a)}{\varphi(\mu)} - 1)$, and weight W is nonincreasing on $[a, b]$, or $E = [b - \varphi(\mu), b]$, $P(x) = AT_n(\frac{2(b-x)}{\varphi(\mu)} - 1)$, and W is nondecreasing on $[a, b]$, $A \in \mathbf{R}^1$.*

Theorem 2.1 allows the following estimates.

COROLLARY 2.2. (a) *For $P \in \mathscr{P}_{n,m}$ and $E \subseteq V$, $|E|_m > 0$,*

$$\|P\|_{C(V)} \leqslant (C_1 \varphi(\mu V)/\varphi(\mu E))^n \|P\|_{C(E)}. \tag{2.14}$$

(b) *For $P \in \mathscr{P}_{n,m}$ and $E \subseteq V$, $1 - 2^{-2m} \leqslant \varphi(\mu E)/\varphi(\mu V) \leqslant 1$,*

$$\|P\|_{C(V)} \leqslant \exp(C_2 n(1 - \varphi(\mu E)/\varphi(\mu V))^{1/2m}) \|P\|_{C(E)}, \tag{2.15}$$

where $C_1 \leqslant 4m$ and $C_2 \leqslant 4$.

Proof. It is easy to show that

$$T_n\left(\frac{1+\beta_m(t)}{1-\beta_m(t)}\right) = (1/2)\left(\left(\frac{1+\beta_{2m}(t)}{1-\beta_{2m}(t)}\right)^n + \left(\frac{1-\beta_{2m}(t)}{1+\beta_{2m}(t)}\right)^n\right), \qquad 0 \leqslant t \leqslant 1.$$

Hence

$$T_n\left(\frac{1+\beta_m(t)}{1-\beta_m(t)}\right) \leqslant \left(\frac{1+(1-t)^{1/(2m)}}{1-(1-t)^{1/(2m)}}\right)^n \leqslant \left(\frac{4m}{t}\right)^n, \qquad 0 < t \leqslant 1,$$

$$T_n\left(\frac{1+\beta_m(t)}{1-\beta_m(t)}\right) \leqslant \left(1 + \frac{2(1-t)^{1/(2m)}}{1-(1-t)^{1/(2m)}}\right)^n \leqslant \exp\left(\frac{2n(1-t)^{1/(2m)}}{1-(1-t)^{1/(2m)}}\right)$$
$$\leqslant \exp(4n(1-t)^{1/(2m)}), \qquad 1 - 2^{-2m} \leqslant t \leqslant 1.$$

Together with (2.1) this yields (2.14) and (2.15). ∎

Remark 2.2. Analogues of inequalities (2.13), (2.14), and (2.15) are also valid for generalized polynomials $f(z) = |C| \prod_{i=1}^k |z - z_i|^{\alpha_i}$, $z_i \in \mathbf{C}$, $\alpha_i > 0$, $1 \leqslant i \leqslant k$ of degree $N = \sum_{i=1}^k \alpha_i$, if we use (2.4) and a Remez-type inequality for f [3, p. 393]:

$$\|f\|_{C(a,b)} \leqslant (\sqrt{(b-a)/|E|_1} + \sqrt{(b-a)/|E|_1 - 1})^{2N} \|f\|_{C(E)},$$
$$E \subseteq [a, b], \qquad |E|_1 > 0. \tag{2.16}$$

The following corollary shows that inequalities (2.1) and (2.15) are equivalent to certain estimates of μ-rearrangements. The corresponding nonweighted results were obtained in [6, 7, 22, 24].

COROLLARY 2.3. (a) *Theorem* 2.1(a) *is equivalent to the following statement*: *for every* $P \in \mathscr{P}_{n,m}$ *and every* $t \in (0, \mu V]$,

$$P_\mu^*(t) \geqslant \left(T_n \left(\frac{1+\beta_m(\varphi(t)/\varphi(\mu V))}{1-\beta_m(\varphi(t)/\varphi(\mu V))} \right) \right)^{-1} \|P\|_{C(V)}. \tag{2.17}$$

(b) *Corollary* 2.2(b) *is equivalent to the statement*: *for every* $P \in \mathscr{P}_{n,m}$ *and every* $t \in (0, \mu V]$ *such that* $1 - 2^{-2m} \leqslant \varphi(t)/\varphi(\mu V) \leqslant 1$,

$$P_\mu^*(t) \geqslant \exp(-4n(1-\varphi(\mu E)/\varphi(\mu V))^{1/2m}) \|P\|_{C(V)}. \tag{2.18}$$

Proof. Let $P \in \mathscr{P}_{n,m}$ and let (2.1) hold for every $E \subseteq V$. Then $\mu E_t = t$ for $t \in (0, \mu V]$ and a set $E_t := \{x \in V : |P(x)| \leqslant P_\mu^*(t)\}$. Hence (2.17) follows from (2.1). Conversely, if $P \in \mathscr{P}_{n,m}$ and (2.17) holds for every $t \in (0, \mu V]$, then for every $E \subseteq V$ and $E' = \{x \in V : |P(x)| \leqslant \|P\|_{C(E)}\}$ we have $\mu E \leqslant \mu E'$ and $\|P\|_{C(E)} = \|P\|_{C(E')} = P_\mu^*(\mu E')$. Now (2.1) follows from (2.17). This establishes statement (a) of the corollary. Similarly statement (b). ∎

Remark 2.3. It easy to verify that for any bounded and measurable sets E and Ω, $E \subseteq \Omega \subset \mathbf{R}^m$, we have $|E|_m/|\Omega|_m \geqslant \varphi(\mu E)/\varphi(\mu V)$ (cf. (2.4)). Hence the following general form of Theorem 2.1 is valid: if for every $P \in \mathscr{P}_{n,m}$,

$$\|P\|_{C(\Omega)} \leqslant \Psi(|E|_m/|V|_m) \|P\|_{C(E)}, \tag{2.19}$$

where Ψ is nonincreasing on $(0, 1]$, then

$$\|P\|_{C(\Omega)} \leqslant \Psi(\varphi(\mu E)/\varphi(\mu V)) \|P\|_{C(E)}. \tag{2.20}$$

Moreover, (2.20) implies (see the proof of Corollary 2.3)

$$P_\mu^*(t) \geqslant (\Psi(\varphi(t)/\varphi(\mu V)))^{-1} \|P\|_{C(\Omega)}, \qquad 0 < t \leqslant \mu V. \tag{2.21}$$

In particular, it is possible to refine inequalities (2.15) and (2.18) for bounded domains $\Omega \subset \mathbf{R}^m$ with C^2-boundaries:

$$\begin{gathered} \|P\|_{C(\Omega)} \leqslant \exp(Cn(1-\varphi(\mu E)/\varphi(\mu\Omega))^{1/(m+1)}) \|P\|_{C(E)}, \\ C \leqslant \varphi(\mu E)/\varphi(\mu\Omega) \leqslant 1, \\ P_\mu^*(t) \geqslant \exp(-Cn(1-\varphi(t)/\varphi(\mu\Omega))^{1/(m+1)}) \|P\|_{C(\Omega)}, \\ C \leqslant \varphi(t)/\varphi(\mu\Omega) \leqslant 1. \end{gathered} \tag{2.22}$$

To prove these relations, we note that (2.19) holds for such domains with $\Psi(y)=\exp(Cn(1-y)^{1/(m+1)})$ (see [31]). It remains to apply (2.20) and (2.21).

It is also possible to extend (2.15) and (2.18) to a bounded domain $\Omega\subset\mathbf{R}^m$ satisfying the cone property [1, p. 66]. Indeed, it is easy to show [24, 31] that (2.19) holds for such a domain and the corresponding Ψ. Then (2.20) and (2.21) yield the corresponding estimates.

2.2. *A Homogeneous Inequalitiy for Algebraic Polynomials*

A constant in the nonweighted version of (2.14) $\|P\|_{C(V)}\leqslant(4m\,|V|_m/|E|_m)^n\,\|P\|_{C(E)}$ depends only on n, m, and $|V|_m/|E|_m$. Such a homogeneity of the constant for fixed n and M plays an important role in some applications [5, 20, 32, 39, 40].

Below we define a condition on a weight W that guarantees the validity of the homogeneous weighted inequality

$$\|P\|_{C(V)}\leqslant(C_3\mu V/\mu E)^n\,\|P\|_{C(E)} \tag{2.23}$$

for every $P\in\mathscr{P}_{n,m}$ and all measurable E and convex V, such that $|E|_m>0$ and $E\subseteq V\subseteq\Omega$, where Ω is a fixed domain in $\mathbf{R}^m$. We also show that in the one-dimensional case the condition is necessary for continuous monotone weights.

Definition 2.1. We say that W satisfies the Δ_M-condition on Ω if there exists a number $M=M(W,\Omega)\in(0,1)$ such that

$$C_0=C_0(M,W,\Omega):=\sup_{V\subseteq\bar{\Omega}} W_V^*(|V|_m)/W_V^*(M\,|V|_m)<\infty. \tag{2.24}$$

Let π be a family of convex bodies $V\subseteq\bar{\Omega}$.

Definition 2.2. We say that W satisfies the $\Delta_M(\pi)$-condition on Ω if there exists $M\in(0,1)$ such that

$$C_0(\pi)=\sup_{V\in\pi} W_V^*(|V|_m)/W_V^*(M\,|V|_m)<\infty.$$

Theorem 2.2. (a) *If W satisfies the Δ_M-condition on Ω, then for a polynomial $P\in\mathscr{P}_{n,m}$, a convex body $V\subseteq\bar{\Omega}$, and an L-measurable set $E\subseteq V$, $|E|_m>0$, inequality (2.23) holds with $1<C_3\leqslant(1-M)^{-1}\,C_0$.*

(b) *If W satisfies the $\Delta_M(\pi)$-condition on Ω, then for every $P\in\mathscr{P}_{n,m}$, $V\in\pi$, and $E\subseteq V$, $|E|_m>0$, (2.23) holds.*

(c) *If W is a nondecreasing (or nonincreasing) continuous weight on $\bar{\Omega}=[A, B]$, and (2.23) holds for $C_3>1$, every $P\in\mathscr{P}_{n,m}$ and all sets $E\subseteq V\subseteq [A, B]$, $|E|_1>0$, then W satisfies the Δ_M-condition on Ω, with $M=1-(4C_3)^{-1}$ and $C_0\leqslant 4C_3-1$.*

Proof. (a) We note first that the relations

$$\mu E=\int_{|V|_m-\varphi(\mu E)}^{|V|_m} W_V^*(\tau)\,d\tau\leqslant\varphi(\mu E)\,W_V^*(|V|_m),$$

$$\mu V=\int_0^{|V|_m} W_V^*(\tau)\,d\tau\geqslant\int_{M|V|_m}^{|V|_m} W_V^*(\tau)\,d\tau\geqslant(1-M)\,\varphi(\mu V)\,W_V^*(M|V|_m)$$

imply the estimates

$$\frac{\varphi(\mu V)}{\varphi(\mu E)}\leqslant(1-M)^{-1}\,\frac{W_V^*(|V|_m)}{W_V^*(M\,|V|_m)}\,\frac{\mu V}{\mu E}\leqslant(1-M)^{-1}\,C_0\,\frac{\mu V}{\mu E}. \tag{2.25}$$

Thus (2.14) and (2.25) yield (2.23).

Statement (b) can be proved similarly.

(c) Without loss of generality we assume that W is nondecreasing on (A, B). Then $W_V^*(\tau)=W(a-A+\tau)$ for any interval $V=[a, b]\subseteq(A, B)$. Next, choosing for a fixed $\mu\in(0, \mu V]$, $E=[a, b-\varphi(\mu)]$ and $P(x)=T_n(2(b-x)/\varphi(\mu)-1)$, we obtain from Corollary 2.1

$$\|P\|_{C(a,b)}=T_n(2\varphi(\mu V)/\varphi(\mu)-1)\,\|P\|_{C(E)}\geqslant(\varphi(\mu V)/2\varphi(\mu))^n\,\|P\|_{C(E)}. \tag{2.26}$$

It follows from (2.23) and (2.26) that for all intervals $[a, b]\subseteq[A, B]$ and all $y\in(0, b-a]$,

$$\frac{\int_0^{b-a} W_{[a,b]}^*(\tau)\,d\tau}{\int_{b-a-y}^{b-a} W_{[a,b]}^*(\tau)\,d\tau}\geqslant\frac{b-a}{2C_3 y}. \tag{2.27}$$

Letting $y\to 0$ in (2.27) and setting $M=1-(4C_3)^{-1}$, we have

$$\begin{aligned}(2C_3)^{-1}\,(b-a)\,W_{[a,b]}^*(b-a)&\\ \leqslant\int_0^{b-a} W_{[a,b]}^*(\tau)\,d\tau&=\int_0^{M(b-a)}+\int_{M(b-a)}^{b-a}\\ \leqslant M(b-a)\,W_{[a,b]}^*(M(b-a))&+(1-M)(b-a)\,W_{[a,b]}^*(b-a).\end{aligned}$$

This yields

$$C_0 = \sup_{[a, b] \subseteq (A, B)} W^*_{[a, b]}(b-a)/W^*_{[a, b]}(M(b-a)) \leqslant 4C_3 - 1.$$

Thus W satisfies the Δ_M-condition on (A, B) for $M = 1-(4C_3)^{-1}$ and $C_0 \leqslant 4C_3 - 1$. ∎

COROLLARY 2.4. *If W satisfies the $\Delta_M(\pi)$-condition on $\Omega \subseteq \mathbf{R}^m$, then for a polynomial $P \in \mathscr{P}_{n,m}$ and a convex body $V \in \pi$,*

$$P^*_{\mu, V}(t) \geqslant (t/(C_3 \mu V))^n \, \|P\|_{C(V)}, \tag{2.28}$$

where $C_3 \leqslant (1-M)^{-1} \, C_0(\pi)$.

The implication (2.23) ⇒ (2.28) can be proved similarly to Corollary 2.3.

Remark 2.4. It is easy to see that if W is nondecreasing on (A, B), then condition (2.24) is equivalent to $C'_0 = \sup_{a \in [A, B], \tau \in [0, B-A]} W_{[a, b]}(a - A + \tau)/W_{[a, b]}(a - A + M\tau) < \infty$.

2.3. *Trigonometric Polynomials*

Following is a weighted Remez-type inequality for trigonometric polynomials of a single variable.

THEOREM 2.3. *For a polynomial $T \in \mathscr{T}_n$ and an L-measurable set $E \subseteq (0, 2\pi]$, the following inequalities hold,*

$$\|T\|_{C(0, 2\pi)} \leqslant (C_4/\varphi(\mu E))^{2n} \, \|T\|_{C(E)}, \tag{2.29}$$

$$\|T\|_{C(0, 2\pi)} \leqslant \exp(C_5 n(2\pi - \varphi(\mu E))) \, \|T\|_{C(E)}, \qquad \varphi(\mu E) > 3\pi/2, \tag{2.30}$$

where C_4 and C_5 are absolute constants.

For $W(x) = 1$ (that is $\varphi(\mu E = |E|_1)$ inequality (2.29) with $C_4 \leqslant 87$ was established by Nazarov [41], while (2.30) was proved by Erdelyi [17]. More precise estimates for the constants in the nonweighted inequalities ($C_4 \leqslant 17$ and $C_5 \leqslant 2$) were obtained in [24].

Proof of Theorem 2.3. A function φ is strictly increasing on $[0, 2\pi]$. Hence

$$|E|_1 = \varphi\left(\int_{2\pi - |E|_1}^{2\pi} W^*(\tau)\, d\tau\right) \geqslant \varphi\left(\int_E W(x)\, dx\right) = \varphi(\mu E). \tag{2.31}$$

Now (2.29) and (2.30) follow from (2.31) and the corresponding nonweighted inequalities [17, 24, 41]. ∎

COROLLARY 2.5. *For $T \in \mathcal{T}_n$ and every $t \in (0, \mu(0, 2\pi)]$,*

$$\begin{aligned} &T^*_\mu(t) \geqslant C_4^{-1}(\varphi(t))^{2n} \|T\|_{C(0, 2\pi)}, \\ &T^*_\mu(t) \geqslant \exp(-C_5 n(2\pi - \varphi(t)) \|T\|_{C(0, 2\pi)}, \qquad \varphi(t) \geqslant 3\pi/2, \end{aligned} \tag{2.32}$$

where C_4 and C_5 are constants in (2.29) *and* (2.30).

The proof of the corollary is similar to that of Corollary 2.3 if we use Theorem 2.3 instead of Theorem 2.1.

3. NIKOLSKII- AND SCHUR-TYPE INEQUALITIES IN WEIGHTED SPACES

3.1. *Algebraic Polynomials*

We present first a Remez–Nikolskii-type inequality in WRI spaces.

THEOREM 3.1. *Let $V \subset \mathbf{R}^m$ be a convex body and let $F(V)$ be a WRI space. Then*

(a) *for a polynomial $P \in \mathcal{P}_{n,m}$, an L-measurable set $E \subseteq V$, $|E|_m > 0$, and any $\alpha \in (0, 1]$,*

$$\|P\|_{C(V)} \leqslant (\psi_F(\alpha\mu E))^{-1} T_n\left(\frac{1+\beta_m(\varphi((1-\alpha)\,\mu E)/\varphi(\mu V))}{1-\beta_m(\varphi((1-\alpha)\,\mu E)/\varphi(\mu V))}\right) \|P\|_{F(E)}, \tag{3.1}$$

(b) *for $P \in \mathcal{P}_{n,m}$, $E \subseteq V$, $|E|_m > 0$, $1 - 2^{-2m} \leqslant \varphi(\mu E)/\varphi(\mu V) \leqslant 1$ and any $\alpha \in (0, 1]$,*

$$\|P\|_{C(V)} \leqslant (\psi_F(\alpha\mu E))^{-1} \exp(4n(1-\varphi((1-\alpha)\,\mu E)/\varphi(\mu V))^{1/2m}) \|P\|_{F(E)}. \tag{3.2}$$

Proof. Taking account of (1.1), we have

$$\|P\|_{F(E)} \geqslant \|P^*_{\mu, V}\|_{\tilde{F}(0, \mu E)} \geqslant \|P^*_{\mu, V}\|_{\tilde{F}((1-\alpha)\,\mu E, \mu E)} \geqslant \psi_F(\alpha\mu E)\, P^*_{\mu, V}((1-\alpha)\,\mu E). \tag{3.3}$$

Hence (3.1) follows from (3.3) and (2.17). Similarly, (3.2) is a consequence of (3.3) and (2.18). ∎

Following is a global version of the Nikolskii inequality for algebraic polynomials in weighted spaces.

COROLLARY 3.1. *Let $P \in \mathscr{P}_{n,m}$ be a polynomial.*

(a) *For a convex body V and a WRI space $F(V)$,*

$$\|P\|_{C(V)} \leqslant 8\left(\psi_F\left(\int_0^{|V|_m/(n+1)^{2m}} W_V^*(\tau)\,d\tau\right)\right)^{-1} \|P\|_{F(V)}. \tag{3.4}$$

(b) *For a bounded domain $\Omega \subset \mathbf{R}^m$ with the C^2-boundary and a WRI space $F(\Omega)$,*

$$\|P\|_{C(\Omega)} \leqslant C\left(\psi_F\left(\int_0^{|\Omega|_m/(n+1)^{m+1}} W_\Omega^*(\tau)\,d\tau\right)\right)^{-1} \|P\|_{F(\Omega)}. \tag{3.5}$$

(c) *In particular, for $0 < p \leqslant q \leqslant \infty$,*

$$\|P\|_{L_{q,W}(V)} \leqslant 8^{1-p/q}\left(\int_0^{|V|_m/(n+1)^{2m}} W_V^*(\tau)\,d\tau\right)^{1/q-1/p} \|P\|_{L_{p,W}(V)}, \tag{3.6}$$

$$\|P\|_{L_{q,W}(\Omega)} \leqslant C\left(\int_0^{|\Omega|_m/(n+1)^{m+1}} W_\Omega^*(\tau)\,d\tau\right)^{1/q-1/p} \|P\|_{L_{p,W}(\Omega)}. \tag{3.7}$$

Proof. We first prove (3.4). Putting $\alpha_n = (\mu V)^{-1}\int_0^{|V|_m/(n+1)^{2m}} W^*(\tau)\,d\tau$, we have

$$\varphi((1-\alpha_n)\,\mu V) = (1-(n+1)^{-2m})\,|V|_m = (1-(n+1)^{-2m})\,\varphi(\mu V).$$

Then using (3.1) for $E = V$ and $\alpha = \alpha_n$, we obtain

$$\|P\|_{C(V)} \leqslant C_n\left(\psi_F\left(\int_0^{|V|_m/(n+1)^{2m}} W^*(\tau)\,d\tau\right)\right)^{-1} \|P\|_{F(V)},$$

where

$$C_n = T_n\left(\frac{1+(n+1)^{-2}}{1-(n+1)^{-2}}\right) \leqslant \left(\frac{1+(n+1)^{-1}}{1-(n+1)^{-1}}\right)^n < e^2.$$

This yields (3.4). Equation (3.5) can be proved similarly if we use the corresponding analogue of (3.2) for domains with C^2-boundary that follows from (2.22). Next, (3.6) and (3.7) follow from (3.4) and (3.5) respectively, by the standard argument [47, p. 236]: for $A > 0$ and $0 < p \leqslant q \leqslant \infty$,

$$\begin{aligned}\|P\|_{C(V)} \leqslant A\,\|P\|_{L_{p,W}(V)} \Rightarrow \|P\|_{L_{q,W}(V)} &\leqslant \|P\|_{C(V)}^{1-p/q}\,\|P\|_{L_{p,W}(V)}^{p/q} \\ &\leqslant A^{1-p/q}\,\|P\|_{L_{p,W}(V)}.\end{aligned} \tag{3.8}$$

This completes the proof of the corollary. ∎

Estimate (3.6) was proved in [23] by using (2.3) and some L_p-inequalities for the rearrangements of functions. Apparently this method is inapplicable to (3.5). Recently a version of (3.6) was independently obtained in [31].

Note that (3.5) and (3.7) can be extended to a bounded domain satisfying the cone property if we use the corresponding version of (2.18) (see Remark 2.3).

We remark that Nikolskii-type inequalities (3.4), (3.5), (3.6), and (3.7) are efficient for weights of power growth (see Examples 5.1, 5.2, and 5.4). However, for some weights (in particular, for exponential weights) we need more precise estimates (see Example 5.3).

COROLLARY 3.2. *If V is a convex body and $F(V)$ is a WRI space, then for a polynomial $P \in \mathscr{P}_{n,m}$,*

$$\|P\|_{C(V)} \leqslant \exp\left(H\left(\ln \psi_F\left(\int_0^{y^{2m}|V|_m} W^*(\tau)\,d\tau\right), 4n\right)\right) \|P\|_{F(V)}, \tag{3.9}$$

where $H(f(y), t) = \inf_{y \in (0,1)} (ty - f(y))$ is the Young-type transformation of f.

In particular, for $0 < p \leqslant q \leqslant \infty$,

$$\|P\|_{L_{q,W}(V)} \leqslant \exp\left((1 - p/q)\, H\left(p^{-1} \ln \int_0^{y^{2m}|V|_m} W^*(\tau)\,d\tau, 4n\right)\right) \|P\|_{L_{p,W}(V)}. \tag{3.10}$$

Proof. To prove (3.9), we first note that for $\alpha = (\mu V)^{-1} \int_0^{y|V|_m} W^*(\tau)\,d\tau$ and every $y \in (0,1)$, $\varphi((1-\alpha)\,\mu V)/\varphi(\mu V) = 1 - y$. Next, applying (3.2) for $E = V$, we obtain

$$\|P\|_{C(V)} \leqslant \exp\left(4ny^{1/2m} - \ln \psi_F\left(\int_0^{y|V|_m} W^*(\tau)\,d\tau\right)\right) \|P\|_{F(V)}.$$

This yields (3.9). Then (3.10) follows from (3.8) and (3.9). ∎

Recall that the estimates similar to (3.9) and (3.10) can be obtained for bounded domains with the cone property or with the C^2-boundary.

The following is a homogeneous version of the Nikolskii inequality.

COROLLARY 3.3. *Let Ω be a domain in $\mathbf{R}^m$ and let $F(\Omega)$ be a NWRI space. If W satisfies the $\Delta_M(\pi)$-condition on Ω, where π is a family of convex bodies $V \subseteq \Omega$, then for a polynomial $P \in \mathscr{P}_{n,m}$ and a convex body $V \in \pi$,*

$$\|P\|_{C(V)} \leqslant C(\psi_F(\mu V))^{-1} \|P\|_{F(V)}, \tag{3.11}$$

where $C \leqslant 4(2C_0(\pi)(1-M)^{-1})^n$ and $C_0(\pi)$ is defined in Definition 2.2.

Proof. Using Corollary 2.4, we obtain

$$\|P\|_{F(V)} \geqslant \|P^*_{\mu, V}\|_{\tilde{F}(\mu V/2, \mu V)} \geqslant P^*_{\mu, V}(\mu V/2)\, \psi_F(\mu V/2)$$
$$\geqslant \psi_F(\mu V/2)(2C_3)^{-n}\, \|P\|_{C(V)}. \tag{3.12}$$

Since by (1.2), $\psi_F(\tau/2) \geqslant (1/4)\, \psi_F(\tau)$, $\tau > 0$, (3.12) yields (3.11). ▮

Finally, we obtain a weighted L_∞-version of (3.4) (a Schur-type inequality).

COROLLARY 3.4. *Let $P \in \mathscr{P}_{n, m}$ be a polynomial and W a continuous weight. Then*

(a) *For a convex body V,*

$$\|P\|_{C(V)} \leqslant 8(W^*_V(|V|_m\,(n+1)^{-2m}))^{-1}\, \|WP\|_{C(V)}. \tag{3.13}$$

(b) *For a bounded domain Ω with C^2-boundary,*

$$\|P\|_{C(\Omega)} \leqslant C(W^*_\Omega(|\Omega|_m\,(n+1)^{-(m+1)}))^{-1}\, \|P\|_{C(\Omega)}. \tag{3.14}$$

Proof. Relations (3.13) and (3.14) follow from (3.6) and (3.7), respectively, if we replace W with W^p in (3.6) and (3.7) for $q = \infty$, take account of $(W^p)^* = W^{*p}$, and let $p \to \infty$. ▮

Remark 3.1. In 1919 Schur [3, 37, 45] established the estimates

$$\|P\|_{C(-1, 1)} \leqslant \min(n\, \|\sqrt{1-x^2}\, P(x)\|_{C(-1, 1)}, (n+1)\, \|xP(x)\|_{C(-1, 1)}),$$
$$P \in \mathscr{P}_{n, 1}. \tag{3.15}$$

Goetgheluck [25] noticed that, by Markov's inequality,

$$\|P\|_{C(-1, 1)} \leqslant (n+1)^2\, \|(1-x)\, P(x)\|_{C(-1, 1)}, \qquad P \in \mathscr{P}_{n, 1}, \tag{3.16}$$

and established [26, 27, 28] the generalized version of (3.15) and (3.16)

$$\|P\|_{L_p(\Omega)} \leqslant Cn^d\, \|WP\|_{L_p(\Omega)}, \qquad P \in \mathscr{P}_{n, m}, \qquad 1 \leqslant p \leqslant \infty, \tag{3.17}$$

where the open bounded set $\Omega \subseteq \mathbf{R}^m$ and a weight $W \in L_1(\Omega)$ satisfy certain conditions, and the exponent d is effectively computable.

Note that for $p = \infty$ and the convex sets or the sets with C^2-boundaries, inequalities (3.13) or (3.14) are more general than (3.17) and give more precise estimates for certain weights. For example, if $V = [0, 1]$ and $W(x) = (\ln(e/x))^{-1}$, then (3.13) yields

$$\|P\|_{C(0, 1)} \leqslant 8(1 + 2\ln(n+1))\, \|P\|_{C(0, 1)}.$$

The similar example can be constructed for a cube or a ball in $\mathbf{R}^m$.

3.2. *Trigonometric Polynomials*

Throughout this section C_5 is the absolute constant in (2.30).

THEOREM 3.2. *If $F(0, 2\pi)$ is a WRI space of 2π-periodic functions of a single variable, then for $T \in \mathcal{T}_n$, $E \subseteq (0, 2\pi]$, $|E|_1 > 0$, and any $\alpha \in (0, 1]$,*

$$\begin{aligned} &\|T\|_{C(0, 2\pi)} \leqslant (\psi_F(\alpha\mu E))^{-1} \exp(C_5 n(2\pi - \varphi((1-\alpha)\,\mu E)))\,\|T\|_{F(E)}, \\ &\quad \varphi(\mu E) > 3\pi/2. \end{aligned} \tag{3.18}$$

Proof. Taking account of (1.1) and (2.32), we obtain

$$\begin{aligned} \|T\|_{F(E)} &\geqslant \|T_\mu^*\|_{\tilde{F}(0, \mu E)} \geqslant \|T_\mu^*\|_{\tilde{F}((1-\alpha)\,\mu E, \mu E)} \geqslant \psi_F(\alpha\mu E)\, T_\mu^*((1-\alpha)\,\mu E) \\ &\geqslant \psi_F(\alpha\mu E) \exp(-C_5 n(2\pi - \varphi((1-\alpha)\,\mu E)))\,\|T\|_{C(0, 2\pi)}. \quad \blacksquare \end{aligned}$$

Then (3.18) follows. ∎

COROLLARY 3.5. *For a WRI space F and a polynomial $T \in \mathcal{T}_n$,*

$$\|T\|_{C(0, 2\pi)} \leqslant e^{2\pi C_5} \left(\psi_F\left(\int_0^{2\pi/(n+1)} W^*(\tau)\, d\tau\right)\right)^{-1} \|T\|_{F(0, 2\pi)},$$

$$\|T\|_{C(0, 2\pi)} \leqslant \exp\left(H\left(\ln \psi_F\left(\int_0^{2\pi y} W^*(\tau)\, d\tau\right), C_5 n\right)\right) \|T\|_{F(0, 2\pi)},$$

where $H(f(y), x)$ is the transformation from Corollary 3.2. *In particular, for $0 < p \leqslant q \leqslant \infty$,*

$$\|T\|_{L_{q,W}(0, 2\pi)} \leqslant e^{2\pi C_5(1-p/q)} \left(\int_0^{2\pi/(n+1)} W^*(\tau)\, d\tau\right)^{1/q - 1/q} \|T\|_{L_{p,W}(0, 2\pi)},$$

$$\|T\|_{L_{q,W}(0, 2\pi)} \leqslant \exp\left((1-p/q)\, H\left(p^{-1} \ln \int_0^{2\pi y} W^*(\tau)\, d\tau, C_5 n\right)\right) \|T\|_{L_{p,W}(0, 2\pi)}.$$

COROLLARY 3.6. *For a polynomial $T \in \mathcal{T}_n$ and a weight $W \in C(0, 2\pi)$,*

$$\|T\|_{C(0, 2\pi)} \leqslant e^{2\pi C_5}(W^*(2\pi/(n+1)))^{-1}\,\|WT\|_{C(0, 2\pi)}. \tag{3.19}$$

The proofs of Corollaries 3.5 and 3.6 are similar to those of Corollaries 3.1, 3.2, and 3.3.

Remark 3.2. A Schur-type inequality

$$\|T\|_{L_p(0, 2\pi)} \leqslant Cn^r\,\|WT\|_{L_p(0, 2\pi)}, \qquad 1 \leqslant p \leqslant \infty, \quad T \in \mathcal{T}_n \tag{3.20}$$

for the generalized Jacobi weight $W(x)=\prod_{i=1}^{k}|x-x_i|^{\alpha_i}$, $\alpha_i>0$, $1\leqslant i\leqslant k$, and $r=\max_{1\leqslant i\leqslant k}\alpha_i$ was proved in [25]. Below we show (see Corollary 5.1(d)) that (3.20) follows from (3.19) for $p=\infty$.

4. AN ESTIMATE OF THE ERROR OF LOCAL BEST APPROXIMATION

4.1. *Statement of Main Results*

Here we consider some applications of the homogeneous Nikolskii-type inequality (see Corollary 3.3) to Jackson-type estimates in approximation theory. We obtain an estimate of the error of best polynomial approximation of a function in a weighted space via its local approximation characteristic. As a corollary, we establish an inequality for the weighted rearrangement of a function from a NWRI space.

To formulate the results, we shall need some definitions.

Let $E(f, B, F) := \inf_{g\in B}\|f-g\|_F$ be the error of best approximation in the metric of a normed space F of $f\in F$ by elements from a subspace $B\subseteq F$. Let $Q_0 := [-1, 1]^m$ be the cube in $\mathbf{R}^m$, and Q a subcube, that is a closed cube in Q_0 that is homothetic to Q_0.

DEFINITION 4.1. A family of disjoint subcubes is called a packing. A packing $\pi_\tau=\{Q\}$ with $\mu Q=\tau$, $0<\tau\leqslant\mu Q_0$, $Q\in\pi_\tau$, is called a (μ, τ)-packing.

Let $F(Q_0)$ be an NWRI space, π_τ a (μ, τ)-packing, and $\mathbf{\Pi}_{\mathbf{n}-\mathbf{1}}(\pi_\tau)$ the class of all piecewise-polynomial functions g such that $g|_Q\in\mathscr{P}_{n-1,m}$ for each $Q\in\pi_\tau$.

We define the local approximation characteristic of $f\in F(Q_0)$ by

$$\omega(f, \tau)=\omega(f, \mathbf{\Pi}_{\mathbf{n}-\mathbf{1}}(\pi_\tau(\pi_\tau), F(Q_0))$$

$$:=\sup_{\pi_\tau} E\left(f, \mathbf{\Pi}_{\mathbf{n}-\mathbf{1}}(\pi_\tau), F\left(\bigcup_{Q\in\pi_\tau} Q\right)\right), \qquad 0<\tau\leqslant\mu Q_0.$$

Set $\omega(f, \tau) := \omega(f, \mu Q_0)$ for $\tau>\mu Q_0$.

THEOREM 4.1. *If W satisfies the $\Delta_M(\kappa)$-condition on Q_0, where κ is a family of all subcubes, then for an L-measurable $E\subseteq Q_0$ and any $f\in F(Q_0)$,*

$$E(f, \mathscr{P}_{n-1,m}, F(E))\leqslant C\psi_F(\mu E)\int_{2\mu E}^{4\mu Q_0}\frac{\omega(f, \tau)}{\psi_F(\tau)}\frac{d\tau}{\tau}, \tag{4.1}$$

where C is independent of f and E.

The following corollary plays an important role in some areas of analysis [4, 5, 20].

COROLLARY 4.1. *If W satisfies the $\Delta_M(\kappa)$-condition on Q_0, then for any $f \in F(Q_0)$, there exists $P_0 \in \mathscr{P}_{n-1,m}$ such that*

$$(f-P_0)_{*\,Q_0}(t) \leqslant C \int_{2t}^{4\mu Q_0} \frac{\omega(f,\tau)}{\psi_F(\tau)} \frac{d\tau}{\tau}, \qquad 0<t\leqslant \mu Q_0. \tag{4.2}$$

The nonweighted versions of Theorem 4.1 and Corollary 4.1 in more general settings were proved by Brudnyi [4, 5]. The author [20] established (4.1) and (4.2) for packings $\pi_\tau = \{Q\}$ with $|Q|_m = \tau$ and weights W satisfying the condition $\inf_{x\in Q_0} W(x) \geqslant C > 0$.

The proofs are the modification of those of Theorem 1 and Corollary 2 in [5] and Lemma 2.1 in [20]. The proof of Theorem 4.1 is based on Corollary 3.3 and a new covering lemma for weighted measures (Lemma 4.2).

4.2. *Covering Lemmas*

LEMMA 4.1. *Let E be a bounded set in $\mathbf{R}^m$, and let for every $x \in E$, there exist a closed interval $\Pi(x)$ satisfying the condition: $B_{r(x)}(x) \subseteq \Pi(x) \subseteq B_{Cr(x)}(x)$, where $B_{r(x)}(x)$ and $B_{Cr(x)}(x)$ are two balls centered at x and $C \geqslant 1$ is a constant independent of x. Then a family $\{\Pi(x)\}_{x\in E}$ contains a sequence $\pi = \{\Pi(x_k)\}_{k=1}^{\infty}$ with the following properties:*

(a) $E \subseteq \bigcup_{k=1}^{\infty} \Pi(x_k)$;

(b) *the multiplicity of the covering of points of $\mathbf{R}^m$ by π does not exceed $\theta_1(m, C)$;*

(c) *there exist packings π_i, $1 \leqslant i \leqslant \theta_2(m, C)$, such that $\pi = \bigcup_{i=1}^{\theta_2} \pi_i$.*

The lemma is a special case of Morse's theorem [29, 38].

The following weighted version of the covering lemma is trivial for $m=1$ or $m>1$, $W(x)=1$. However, the author could not find a short proof in the general case.

LEMMA 4.2. *For a weight $W \in L_1(Q_0)$ and a fixed $\tau \in (0, \mu Q_0)$, there exists a family $\pi = \{Q\}$ of subcubes with the following properties:*

(a) $\mu Q = \tau$ *for all* $Q \in \pi$;

(b) $Q_0 = \bigcup_{Q\in\pi} Q$;

(c) *the multiplicity of the covering of points of $\mathbf{R}^m$ by π does not exceed $\theta_3(m)$;*

(d) *there exist packings π_i, $1 \leqslant i \leqslant \theta_4(m)$, such that $\pi = \bigcup_{i=1}^{\theta_4} \pi_i$.*

Proof. We construct first a special family of subcubes with properties (a) and (b). Let $Q_0 = A' \cup A''$, where A' is a set of points from Q_0 such that for each $x \in A'$ there exists a subcube $Q(x) \subseteq Q_0$, $\mu Q = \tau$, centered at x, and $A'' = Q_0 - A'$. Set $\pi' = \{Q(x) : x \in A'\}$.

To select a subcube, covering a point $x = (x_1, ..., x_m) \in A''$, we shall need a special polygon curve $c(x)$. We assume first that $x_1 \geqslant \cdots \geqslant x_m \geqslant 0$. Put

$$A_i = (x_i, ..., x_i, x_{i+1}, ..., x_m) \in \mathbf{R}^m, \qquad 1 \leqslant i \leqslant m; \quad A_{m+1} = (0, ..., 0).$$

Then the curve is defined by $c(x) = \bigcup_{i=1}^m J_i$, where J_i is a segment with starting point A_i and ending point A_{i+1}, $1 \leqslant i \leqslant m$. Note that $J_i = J_{i+1}$ if $x_i = x_{i+1}$, $1 \leqslant i \leqslant m$.

The following properties of $c(x)$ are valid.

(a) $c(x)$ is a simple curve in Q_0 with starting point x and ending point 0.

(b) There exists $y = y(x) \in c(x)$ such that a subcube $Q(y)$ of the edge length $r(y) = 2\rho(y, \partial Q_0)$ satisfies the condition $\mu Q(y) = \tau$.

(c) The edge length l of $Q(y)$ is equal to $2 - 2(tx_s + (1-t)\, x_{s+1})$, where $y \in J_s$, that is $y = tA_s + (1-t)\, A_{s+1}$ for some s, $1 \leqslant s \leqslant m$ and some $t \in [0, 1]$.

(d) $x \in Q(y)$.

The curve is not self-intersecting since a jth coordinate of A_i is a nonincreasing sequence of numbers for each fixed j, $1 \leqslant j \leqslant m$ and $i = 1, ..., m+1$. Hence (a) follows.

Property (b) is a consequence of (a) and the Weierstrass theorem since $\mu Q(y)$ is a continuous function of $y \in c(x)$, $\mu Q(A_1) = \mu Q(x) < \tau$ (we remind that $x \in A''$), and $\mu Q(A_{m+1}) = \mu Q(0) = \mu Q_0 \geqslant \tau$.

Property (c) is obvious while (d) follows from (c) and the relations

$$0 \leqslant x_j - y_j \leqslant l/2 = 1 - (tx_s + (1-t)\, x_{s+1}),$$

$$1 \leqslant j \leqslant s; \quad x_j = y_j, \quad s+1 \leqslant j \leqslant m.$$

It is easy to show that the curve $c(x)$ with properties (a)–(d) can be constructed for every $x \in Q_0$. Thus the family $\pi' \cup \{Q(y); x \in A''\}$ satisfies properties (a) and (b) of the lemma. However, we cannot apply the Besicovich-type theorem [29] to this family since x could be located on or near the boundary of $Q(y)$ if $x \in A''$. That is why we need a special family of intervals that contain $x \in A''$.

Let $x \in A''$, $x_1 \geqslant \cdots \geqslant x_m \geqslant 0$, and let $y = tA_s + (1-t)\, A_{s+1}$, by properties (b) and (c). Set $u = u(x) = (1, ..., 1, x_{s+1}, ..., x_m)$. We define an interval centered at u, by

$$\Pi(u) = \{z \in \mathbf{R}^m : |z_j - u_j| \leqslant l,\ 1 \leqslant j \leqslant s;\ |z_j - u_j| \leqslant l/2,\ s+1 \leqslant j \leqslant m\},$$

where l is the edge length of $Q(y)$, given in property (c). The following properties of $\Pi(u)$ hold.

(e) $Q(y) = \Pi(u) \cap Q_0$.

(f) $x \in \Pi(u)$.

(g) $l/2 \leqslant \rho(x, \partial \Pi(u)) \leqslant \max_{z \in \Pi(u)} |x - z| \leqslant m^{1/2} l$.

To prove property (e), we first note that

$$u - y = (l/2, ..., l/2, 0, ..., 0). \tag{4.3}$$

Hence $u \in Q(y)$ and for every $z \in Q(y)$, $|u_j - z_j| = |y_j - z_j| \leqslant l/2$, $s+1 \leqslant j \leqslant m$. Then it follows from (4.3) that $|u_j - z_j| \leqslant |u_j - y_j| + |y_j - z_j| \leqslant l$, $1 \leqslant j \leqslant s$. Therefore, $Q(y) \subseteq \Pi(u) \cap Q_0$.

It remains to show that if $z \in \Pi(u) - Q(y)$, then $z \notin Q_0$. Indeed, taking account of (4.3) we obtain that the inequalities $|u_j - z_j| \leqslant l$, $|z_j - y_j| > l/2$, are equivalent to $u_j < z_j \leqslant l + u_j$, for each fixed j, $1 \leqslant j \leqslant s$. Hence $z \notin Q_0$, and property (e) follows. Next, note that (f) is an easy consequence of (d) and (e).

Further, the upper estimate in property (g) immediately follows from (f) and the definition of $\Pi(u)$.

To establish the lower estimate, we consider two $(m-1)$-dimensional faces G_{-j} and G_{+j} of $\Pi(u)$ that are orthogonal to the jth coordinate axis, $1 \leqslant j \leqslant m$. If $1 \leqslant j \leqslant s$, then by (c),

$$\rho(x, G_{+j}) = |x_j - (1+l)| = 1 + l - x_j \geqslant l,$$

$$\rho(x, G_{-j}) = |x_j - (1-l)| = x_j - (tx_s + (1-t)\, x_{s+1}) + l/2 \geqslant x_j - x_s + l/2 \geqslant l/2.$$

It is clear that for $s+1 \leqslant j \leqslant m$,

$$\rho(x, G_{\pm j}) = |x_j - (x_j \pm l/2)| = l/2.$$

This proves the lower estimate in (g). The property is established.

Thus properties (b), (d), (e), (f), and (g) show that for every $x \in A''$ satisfying the condition $x_1 \geqslant \cdots \geqslant x_m \geqslant 0$, there exist a subcube $Q(y) \subseteq Q_0$ and an interval $\Pi(u)$ such that

$$\begin{gathered} \mu Q(y) = \tau, \qquad x \in Q(y), \qquad x \in \Pi(u), \qquad Q(y) = \Pi(u) \cap Q_0, \\ B_{l/2}(x) \subseteq \Pi(u) \subseteq B_{m^{1/2} l}(x) \end{gathered} \tag{4.4}$$

for some $l = l(x) > 0$. It is easy to show that (4.4) holds for every $x \in A''$.

Set $\pi'' = \{\Pi(u(x)) : x \in A''\}$ and $\pi^* = \pi' \cup \pi''$. Next, (4.4) shows that for every $x \in Q_0$, there exists a cube or an interval from π^*, satisfying the condition of Lemma 4.1 for $C \leqslant 2m^{1/2}$. Applying now Lemma 4.1, we obtain a family $\pi^{**} = \{\Pi_1, \Pi_2, ...\}$ of subcubes and intervals, satisfying properties (a), (b), and (c) of Lemma 4.1. Then $\pi = \{\Pi_1 \cap Q_0, \Pi_2 \cap Q_0, ...\}$ satisfies all properties of Lemma 4.2. ▮

4.3. *Proofs*

Proof of Theorem 4.1. Let $\pi = \{Q\}$ be a family of subcubes with properties (a), (b), (c), and (d) from Lemma 4.2. According to properties (a) and (d) there exist (μ, τ)-packings π_i, $1 \leqslant i \leqslant \theta_4$, such that $\pi = \bigcup_{i=1}^{\theta_4} \pi_i$. Next, let $P_0 \in \mathscr{P}_{n-1, m}$ be a polynomial of best approximation to f in $F(Q_0)$, and $h_i \in \mathbf{\Pi_{n-1}}(\pi_i)$ a piecewise polynomial function that deviates least from f in $F(\Omega_i)$, where $\Omega_i := \bigcup_{Q \in \pi_i} Q$, $1 \leqslant i \leqslant \theta_4$.

We define $h_i(x) := 0$ for $x \in Q_0 - \Omega_i$, $1 \leqslant i \leqslant \theta_4$, and let $\chi := \sum_{i=1}^{\theta_4} \chi_{\Omega_i}$. Property (b) of Lemma 4.2 shows that $1 \leqslant \chi(x) \leqslant \theta_4$. We set $h = (1/\chi) \times \sum_{i=1}^{\theta_4} h_i \chi_{\Omega_i}$.

Then for any L-measurable $E \subseteq Q_0$, we choose $\tau = \mu E$ and obtain

$$E(f, \mathscr{P}_{n-1, m}, F(E)) \leqslant \|f - P_0\|_{F(E)} \leqslant \|f - h\|_{F(E)} + \|h - P_0\|_{F(E)}. \tag{4.5}$$

Next, by the definition of $\omega(f, \tau)$ and h,

$$\|f - h\|_{F(E)} \leqslant \|f - h\|_{F(Q_0)} \leqslant \sum_{i=1}^{\theta_4} \|f - h_i\|_{F(\Omega_i)} \leqslant C\omega(f, \mu E). \tag{4.6}$$

Similarly,

$$\|h - P_0\|_{F(E)} \leqslant \psi_F(\mu E) \sup_{Q_0} |h - P_0| \leqslant \psi_F(\mu E) \sum_{i=1}^{\theta_4} \sup_{\Omega_i} |h_i - P_0|. \tag{4.7}$$

To estimate

$$\sup_{\Omega_i} |h_i - P_0| = \max_{Q \in \Omega_i} \|h_i - P_0\|_{C(Q)}, \qquad 1 \leqslant i \leqslant \theta_4, \tag{4.8}$$

it suffices to bound $I = \|h_i - P_0\|_{C(Q)}$ from above for every $Q \in \pi_i$, $1 \leqslant i \leqslant \theta_4$.

For the given $Q \in \pi$, $1 \leqslant i \leqslant \theta_4$, $\mu Q = \mu E$, we first define subcubes Q_s such that $\mu Q_s = 2^{s_0 - s} \mu E$, $1 \leqslant s \leqslant s_0$, and $Q = Q_{s_0} \subseteq Q_{s_0 - 1} \subseteq \cdots \subseteq Q_1 \subseteq Q_0$, where $s_0 = [\log_2 (\mu Q_0 / \mu E)] + 1$.

Next, let $P_s \in \mathscr{P}_{n-1,m}$ be the polynomial that deviates least from f in $F(Q_s)$, $1 \leqslant s \leqslant s_0$. Then

$$\|h_i - P_0\|_{C(Q)} \leqslant \|h_i - P_{s_0}\|_{C(Q)} + \sum_{s=0}^{s_0-1} \|P_{s_0-s} - P_{s_0-s-1}\|_{C(Q_{s_0-s})}. \tag{4.9}$$

Further, $h_i|_Q \in \mathscr{P}_{n-1,m}$ and W satisfies the $\Delta(\kappa)$-condition on Q_0, hence applying Corollary 3.3 to polynomials $h_i|_Q - P_{s_0}$ and $P_{s_0-s} - P_{s_0-s-1}$, $1 \leqslant s \leqslant s_0 - 1$, we obtain from (4.9)

$$\begin{aligned}
\|h_i - P_0\|_{C(Q)} &\leqslant C\left(\frac{\|h_i - P_{s_0}\|_{F(Q)}}{\psi_F(\mu E)} + \sum_{s=0}^{s_0-1} \frac{\|P_{s_0-s} - P_{s_0-s-1}\|_{F(Q_{s_0-s})}}{\psi_F(2^s \mu E)}\right) \\
&\leqslant C\left(\frac{\|f - h_i\|_{F(\Omega_i)} + \|f - P_{s_0}\|_{F(Q)}}{\psi_F(\mu E)}\right. \\
&\qquad \left. + \sum_{s=0}^{s_0-1} \frac{\|f - P_{s_0-s}\|_{F(Q_{s_0-s})} + \|f - P_{s_0-s-1}\|_{F(Q_{s_0-s-1})}}{\psi_F(2^s \mu E)}\right) \\
&\leqslant C\left(\sum_{s=0}^{s_0-1} \frac{\omega(f, 2^{s+1}\mu E)}{\psi_F(2^s \mu E)}\right).
\end{aligned} \tag{4.10}$$

Combining (4.5), (4.6), (4.7), (4.8), (4.10) and using (1.2), we have

$$\begin{aligned}
E(f, \mathscr{P}_{n-1,m}, F(E)) &\leqslant C\psi_F(\mu E) \sum_{s=0}^{s_0-1} \frac{\omega(f, 2^{s+1}\mu E)}{\psi_F(2^s \mu E)} \\
&\leqslant 8C\psi_F(\mu E) \int_{2\mu E}^{4\mu Q_0} \frac{\omega(f, \tau)}{\psi_F(\tau)} \frac{d\tau}{\tau}.
\end{aligned}$$

This yields (4.1). ∎

Proof of Corollary 4.1. Let P_0 be the polynomial of best approximation to f in $F(Q_0)$. For a given $t \in (0, \mu Q_0]$, set $E = \{x \in Q_0 : |(f - P_0)(x)| \geqslant (f - P_0)_{*Q_0}(t)\}$. Then $\mu E = t$ and $\|f - P_0\|_{F(E)} \geqslant \psi_F(t)(f - P_0)_{*Q_0}(t)$. Thus (4.2) follows from (4.1). ∎

5. EXAMPLES

5.1. *A Generalized Jacobi Weight*

Let $V = [a, b]$ and let $W(x) = \prod_{i=1}^{k} |x - x_i|^{\alpha_i}$ be a generalized Jacobi weight [3, 18, 25], where $x_i \in \mathbf{R}^1$, $\alpha_i > 0$, and $x_i \neq x_j$ if $i \neq j$, $1 \leqslant i, j \leqslant k$.

THEOREM 5.1. (a) *If* $x_i \in [a, b]$, $1 \leqslant i \leqslant k$, *then for every* $\tau \in (0, \mu V]$,

$$C_6 \tau^d \leqslant W^*_{[a,b]}(\tau) \leqslant C_7 \tau^d, \tag{5.1}$$

where $d = \max_{1 \leqslant i \leqslant k} \alpha_i$ *and* C_6, C_7 *are constants independent of* τ.

(b) W *satisfies the* Δ_M*-condition on* $\mathbf{R}^1$ *for every* $M \in (0, 1)$ *and* $C_0 \leqslant (4/M)^N$, *where* $N = \sum_{i=1}^k \alpha_i$.

To prove the theorem, we shall need a limit version of the Remez inequality for W.

LEMMA 5.1. *If* $x_i \in [a, b]$, $1 \leqslant i \leqslant k$, *then the following relations hold*

$$C_8 H(W) \leqslant \liminf_{t \to 0} t^{-d} W^*_{[a,b]}(t) \leqslant \limsup_{t \to 0} t^{-d} W^*_{[a,b]}(t) \leqslant C_9 H(W), \tag{5.2}$$

where $d = \max_{1 \leqslant i \leqslant k} \alpha_i$, $C_8 = 1/\max(1, (2k)^{d-1})$, $C_9 = \max(1, (2k)^{d-1})$, *and for* $1 \leqslant j \leqslant k$,

$$H(W) = \left(\sum_{j=1}^{k} \delta_j \gamma_j \prod_{i=1,\, i \neq j}^{k} |x_i - x_j|^{-\alpha_i} \right)^{-1},$$

$$\delta_j = \begin{cases} 1, & (x_j - a)(x_j - b) = 0 \\ 2, & x_j \in (a, b), \end{cases} \qquad \gamma_j = \begin{cases} 0, & \alpha_j < d \\ 1, & \alpha_j = d. \end{cases}$$

Proof. We may assume that $a \leqslant x_1 < \cdots < x_k \leqslant b$. If t is small enough, then for every $x_j \in (a, b)$, there exist two numbers y_j^- and y_j^+ such that $y_j^- < x_j < y_j^+$, $W(y_j^-) = W(y_j^+) = W^*_{[a,b]}(t)$ and W is decreasing on $[y_j^-, x_j]$ and increasing on $[x_j, y_j^+]$. The corresponding one-side conditions hold if $x_1 = a$ or (and) $x_k = b$. Further

$$t = \varepsilon_1 (x_1 - y_1^-) + \varepsilon_2 (y_k^+ - x_k) + \sum_{j=2}^{k-1} (y_j^+ - y_j^-) + (y_1^+ - x_1) + (x_k - y_k^-),$$

where $\varepsilon_1 = \begin{cases} 0, & x_1 = a \\ 1, & x_1 > a \end{cases}$ and $\varepsilon_2 = \begin{cases} 0, & x_k = b \\ 1, & x_k < b \end{cases}$. Hence

$$\begin{aligned} \liminf_{t \to 0} t^{-d} W^*(t) \geqslant C_8 \lim_{t \to 0} \Bigg(& \varepsilon_1 / (W^*(t)/(x_1 - y_1^-)^d) + \varepsilon_2 / (W^*(t)/(y_k^+ - x_k)^d) \\ & + \sum_{j=1}^{k-1} 1/(W^*(t)/(y_j^+ - x_j)^d \\ & + \sum_{j=2}^{k} 1/(W^*(t)/(x_j - y_j^-)^d \Bigg)^{-1}. \end{aligned} \tag{5.3}$$

Next,

$$\lim_{t\to 0}\frac{W^*(t)}{(y_j^+-x_j)^d}=\lim_{y_j^+\to x_j}\frac{W(y_j^+)}{(y_j^+-x_j)^d}=\begin{cases}\infty, & \alpha_j<d\\ \prod_{i=1,\, i\neq j}^{k}|x_j-x_i|^{\alpha_i}, & \alpha_j=d.\end{cases} \tag{5.4}$$

It is easy to verify that the similar relations hold for all terms in (5.3). Thus the lower estimate in (5.2) follows from (5.3) and (5.4). Similarly the upper estimate. ∎

Proof of Theorem 5.1. Note first that (5.1) immediately follows from Lemma 5.1. To prove statement (b), we apply a Remez-type inequality (2.16) to the generalized polynomial W of degree $N=\sum_{i=1}^k \alpha_i$. It implies

$$\|W\|_{C(a,b)}\leqslant 2^{2N}((b-a)/|E|_1)^N\,\|W\|_{C(E)} \tag{5.5}$$

for a set $E\subseteq[a,b]$, $|E|_1>0$. Setting $E=\{x\in[a,b]:W(x)\leqslant W^*_{[a,b]}(t)\}$, $0<t\leqslant b-a$, we obtain $|E|_1=t$ and $\|W\|_{C(E)}=W^*_{[a,b]}(t)$. Then (5.5) yields

$$W^*_{[a,b]}(b-a)/W^*_{[a,b]}(t)=\|W\|_{C(a,b)}/W^*_{[a,b]}(t)\leqslant 2^{2N}((b-a)/t)^N. \tag{5.6}$$

Thus (5.6) shows that for every $M\in(0,1)$,

$$\sup_{-\infty<a<b<\infty} W^*_{[a,b]}(M(b-a))\leqslant 2^{2N}M^{-N}.$$

This completes the proof of the theorem. ∎

The following corollary is a consequence of Corollaries 3.1, 3.5, 3.6 and Theorems 2.2, 5.1.

COROLLARY 5.1. (a) *If W is a generalized Jacobi weight and $F(a,b)$ a WRI space, then for a polynomial $P\in\mathscr{P}_{n,1}$,*

$$\|P\|_{C(a,b)}\leqslant C(\psi_F((b-a)\,n^{-2(d+1)}))^{-1}\,\|P\|_{F(a,b)},$$

where $d=\max_{1\leqslant i\leqslant k}\alpha_i$.

In particular, for $0<p\leqslant q\leqslant\infty$,

$$\|P\|_{L_{q,W}(a,b)}\leqslant Cn^{2(d+1)(1/p-1/q)}\,\|P\|_{L_{p,W}(a,b)}. \tag{5.7}$$

(b) *For an interval $V=[a,b]$, a polynomial $P\in\mathscr{P}_{n,1}$, and a set $E\subseteq[a,b]$, $|E|_1>0$,*

$$\|P\|_{C(a,b)}\leqslant 2^{1+\sum_{i=1}^k\alpha_i}(\mu V/\mu E)^n\,\|P\|_{C(E)}.$$

(c) *For a trigonometric polynomial* $T \in \mathcal{T}_n$ *and a WRI space* $F(0, 2\pi)$,

$$\|T\|_{C(0, 2\pi)} \leqslant C(\psi_F(2\pi n^{-d-1}))^{-1} \|T\|_{F(0, 2\pi)}.$$

In particular, for $0 < p \leqslant q \leqslant \infty$,

$$\|T\|_{L_{q, W}(0, 2\pi)} \leqslant C n^{(d+1)(1/p - 1/q)} \|T\|_{L_{p, W}(0, 2\pi)}.$$

(d) *For* $T \in \mathcal{T}_n$,

$$\|T\|_{C(0, 2\pi)} \leqslant C n^d \|WT\|_{C(0, 2\pi)}.$$

Note that the constant C in Corollary 5.1 is independent of n, P, and T. An L_p-version of Corollary 5.1(d) was obtained in [25].

5.2. *A Jacobi Weight*

Let $V = [-1, 1]$ and $W(x) = w_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha > 0$, $\beta > 0$. It is easy to compute $w^*_{\alpha, \beta}(\tau)$ and $\varphi(t)$ for some α and β. For example,

$$w^*_{\lambda, 0}(\tau) = w^*_{0, \lambda}(\tau) = \tau^\lambda, \qquad \varphi(t)/\varphi(\mu_{\lambda, 0}) = 1 - (1 - t(1+\lambda)\, 2^{-(1+\lambda)})^{1/(1+\lambda)},$$

$$w^*_{\lambda, \lambda}(\tau) = 4^{-\lambda} \tau^\lambda (4 - \tau)^\lambda,$$

where $\lambda \geqslant 0$ and $\mu_{\alpha, \beta} = \mu V = 2^{\alpha+\beta+1} \Gamma(1+\alpha)\, \Gamma(1+\beta)(\Gamma(\alpha+\beta+2))^{-1}$.

For any $\alpha > 0$ and $\beta > 0$, we can give only some estimates of $w^*_{\alpha, \beta}$ and obtain certain polynomial inequalities for $P \in \mathcal{P}_{n, m}$, by Theorem 5.1(a) and Corollaries 3.1 and 3.4(a),

$$C_{10} \tau^{\max(\alpha, \beta)} \leqslant w^*_{\alpha, \beta}(\tau) \leqslant C_{11} \tau^{\max(\alpha, \beta)},$$

$$\|P\|_{L_{q, w_{\alpha, \beta}}(-1, 1)} \leqslant C n^{2(\max(\alpha, \beta)+1)(1/p - 1/q)} \|P\|_{L_{p, w_{\alpha, \beta}}(-1, 1)}, \tag{5.8}$$

$$\|P\|_{C(-1, 1)} \leqslant C n^{2 \max(\alpha, \beta)} \|w_{\alpha, \beta} P\|_{C(-1, 1)}. \tag{5.9}$$

Note that a more general version of (5.8) was established by Daugavet and Rafalson [13], while (5.9) in a more general setting was obtained by Goetgheluck [25].

5.3. *An Exponential Weight*

Let $V = [0, 1]$, $W(x) = x^\lambda \exp(-x^{-\alpha})$, where $\lambda \geqslant 0$ and $\alpha > 0$.

COROLLARY 5.2. (a) *For* $P \in \mathcal{P}_{n, 1}$ *and* $p > 0$,

$$\|P\|_{C(0, 1)} \leqslant C n^{2(\lambda+\alpha+1)/(2\alpha+1)\, p} \times \exp(C_{12}(\alpha)\, p^{-1/(2\alpha+1)} n^{2\alpha/(2\alpha+1)}) \|P\|_{L_{p, W}(0, 1)}. \tag{5.10}$$

(b) *The inequality* $\|P\|_{C(a,b)} \leqslant C(\mu V/\mu E)^n \|P\|_{C(E)}$ *does not hold for all polynomials* $P \in \mathscr{P}_{n,1}$, *all intervals* $[a, b] \subseteq [0, 1]$, *and all L-measurable sets* $E \subseteq [a, b]$, $|E|_1 > 0$.

Proof. First we need some technical estimates. It is clear that $W^*_{[0,1]}(\tau) = W(\tau)$. Next, using the known asymptotic relation [10, p. 14], we obtain

$$\int_0^{y^2} W(\tau)\,d\tau = \alpha^{-1} y^{2(\lambda+\alpha+1)} \exp(-y^{-2\alpha})(1 + O(y^{2\alpha})), \qquad y \to 0. \tag{5.11}$$

Hence

$$\begin{aligned}\exp\left(H\left(p^{-1}\ln\int_0^{y^2} W^*_{[0,1]}(\tau)\,d\tau, 4n\right)\right) &\leqslant \left(\int_0^{y_0^2} W(\tau)\,d\tau\right)^{-1/p} \exp(4ny_0)\\ &\leqslant Cn^{2(\lambda+\alpha+1)/(2\alpha+1)p} \exp(C_{12}(\alpha)\,p^{-1/(2\alpha+1)} n^{2\alpha/(2\alpha+1)}),\end{aligned} \tag{5.12}$$

where $y_0 = (2np/\alpha)^{-1/(2\alpha+1)}$ and $H(f, t)$ is defined in Corollary 3.2. Thus, (5.10) follows from (3.10) and (5.12).

Further W does not satisfy the Δ_M-condition on (0, 1). Indeed,

$$C_0' = \sup_{a,\tau \in [0,1]} W(a+\tau)/W(a+M\tau) \geqslant \sup_{\tau \in (0,1)} W(\tau)/W(M\tau) = \infty \tag{5.13}$$

for any $M \in (0, 1)$. Thus statement (b) follows from (5.13), Remark 2.4, and Theorem 2.2(c). ∎

Nikolskii-type inequalities of the form $\|WP\|_{L_q(-1,1)} \leqslant a(n)\|WP\|_{L_p(-1,1)}$, where $W(x) = \exp(-Q(x))$ is an exponential weight, were established by Lubinsky and Saff [36, 44].

Remark 5.1. Corollary 3.2 is more efficient for the exponential weight than Corollary 3.1 since the application of (3.4) and (5.11) to this weight one gives the estimate $\|P\|_{C(0,1)} \leqslant Cn^{2(\lambda+\alpha+1)/p} \exp(n^{2\alpha}/p)\|P\|_{L_{p,W}(0,1)}$ with the less precise constant than in (5.10).

5.4. *A Generalized Gegenbauer–Timan Weight in* $\mathbf{R}^m$

Let $V = Q_0 = [-1, 1]^m$, $W(x) = W_{\alpha,\beta,m}(x) = (\rho(x, \Gamma))^\alpha (\rho(x, \Gamma) + n^{-2m})^\beta$, where $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\Gamma \subset Q_0$ is an s-dimensional surface of class C^1, $0 \leqslant s \leqslant m-1$. For $\Gamma = \{-1, 1\}$, $W_{\alpha,0,1}$ is equivalent to the Gegenbauer weight $w_{\alpha,\alpha}$, and $W_{0,\beta,1}$ is equivalent to $(\sqrt{1-x^2} + 1/n)^{2\beta}$. The latter has become very popular in approximation theory after Timan and Dzyadyk

[16, 47]. Various weighted estimates for polynomials, including Nikolskii-type inequalities, were proved for $W_{\alpha,0,1}$ in [13, 33], for $W_{0,\beta,1}$ in [14, 33], for $W_{0,\beta,m}$, $\Gamma=\partial\Omega$, in [11, 12], and for $W_{\alpha,\beta,m}$, $\Gamma=\partial V$, in [23].

THEOREM 5.2. (a) *For every* $\tau\in(0,\mu Q_0]$,

$$W^*_{Q_0}(\tau)\geqslant C\begin{cases}\tau^{(\alpha+\beta)/(m-s)}, & 0\leqslant\tau<n^{-2m(m-s)}\\ \tau^{\alpha/(m-s)}n^{-2m\beta}, & \tau\geqslant n^{-2m(m-s)}.\end{cases}\tag{5.14}$$

(b) $W_{\alpha,0,m}$ *satisfies the* $\Delta_M(\kappa)$*-condition on* Q_0 *for every* $M\in(0,1)$, *where* κ *is a family of all subcubes in* Q_0 *(that is the closed cubes in* Q_0 *which are homothetic to* Q_0*).*

To prove the theorem, we shall need three geometric lemmas.

LEMMA 5.2. *Let* $\Gamma\subset\mathbf{R}^m$ *be a compact s-dimensional surface of class* C^1, $0\leqslant s\leqslant m-1$, *and let* $Q(x_0)$ *be a cube in* $\mathbf{R}^m$ *with the following properties:* $x_0\in\Gamma$; $Q(x_0)$ *is homothetic to* Q_0; *and* $|Q(x_0)|_m=t^m$, *where* $0<t\leqslant\delta(\Gamma)$. *Then the following relations hold*

$$t^s\leqslant|Q(x_0)\cap\Gamma|_s\leqslant C(\Gamma)\,t^s.\tag{5.15}$$

Proof. We note first that for each $x\in\Gamma$ there exists a cube $Q(x,\Gamma)$ in $\mathbf{R}^m$ with the edge length $r=r(x)>0$ satisfying the following properties: $Q(x,\Gamma)$ is homothetic to Q_0 and centered at x, and there is an s-dimensional face of $Q(x,\Gamma)$ (denote it by $Q^{(s)}(x)$) such that the projection $\mathscr{P}$: $Q(x,\Gamma)\cap\Gamma\to Q^{(s)}(x)$ is a diffeomorphism. Without loss of generality we may assume that $Q(x,\Gamma)\cap\Gamma$ in a neighbourhood of x is determined by the equations $x_i=f_i(x_1,...,x_s)$, where f_i, $s+1\leqslant i\leqslant m$, are smooth functions and $x^{(s)}=(x_1,...,x_s)\in Q^{(s)}(x)$. Then $|Q(x,\Gamma)\cap\Gamma|_s=\int_{Q^{(s)}(x)}|\mathscr{P}'(x^{(s)})|\,dx^{(s)}$, where $|\mathscr{P}'|$ (the modulus of the derivative of $\mathscr{P}$) is continuous on $Q^{(s)}(x)$ [46, p. 327]. Hence

$$1\leqslant|\mathscr{P}'(x^{(s)})|\leqslant C(x)<\infty,\qquad x^{(s)}\in Q^{(s)}(x).\tag{5.16}$$

Next, a family $\{Q_{r/2}(x)\}_{x\in\Gamma}$ is a covering of Γ, and it is possible to choose a finite subcovering $\{Q_{r_j/2}(x^j)\}_{j=1}^N$. Set

$$\delta(\Gamma)=\min_{1\leqslant j\leqslant N}r_j/2,\qquad C(\Gamma)=\max_{1\leqslant j\leqslant N}C(x^j).\tag{5.17}$$

Let $Q(x_0)$ satisfy the properties of the lemma. Then there exists a cube $Q_{r_j/2}(x^j)$ from the subcovering such that $Q_{r_j/2}(x^j)\cap Q(x_0)\neq\varnothing$. Hence

$Q(x_0) \subseteq Q_{r_j}(x^j)$, by (5.17). Moreover, $|Q(x_0) \cap \Gamma|_s = \int_{Q^{(s)}(x_0)} |\mathscr{P}'(x^{(s)}| \, dx^{(s)}$, where $Q^{(s)}(x_0)$ is an s-dimensional subcube of $Q_{r_j}(x^j)$ with the edge length t. Thus (5.15) follows from (5.16) and (5.17). ∎

LEMMA 5.3. *Let $\pi = \{Q_t(x_j)\}$ be a sequence of cubes in $\mathbf{R}^m$ with the edge length t such that the multiplicity of the covering of each point of $\mathbf{R}^m$ by the cubes from π does not exceed N. Then the multiplicity of the covering of points of $\mathbf{R}^m$ by the family $\pi(\gamma) = \{Q_{\gamma t}(x_j)\}$, where $\gamma > 1$, does not exceed $(\gamma+1)^m \, (2^{m-1}(N-1)+1)$.*

Proof. It follows from a result of Brudnyi and Kotlyar [8] that π can be represented in the form $\pi = \bigcup_{l=1}^{L} \pi_l$, where $L = 2^{m-1}(N-1)+1$ and π_l, $1 \leqslant l \leqslant L$, are packings (see Definition 4.1).

It is easy to show that if a point $x \in \mathbf{R}^m$ belongs to N_l cubes $Q_{\gamma t}(y_i)$ from $\pi_l(\gamma)$, then each of these cubes $Q_t(y_i)$ is a subset of $Q_{(\gamma+1)\,t}(x)$. Taking into accout that they are mutually disjoint, we obtain $N_l \leqslant (\gamma+1)^m$ (note that the estimate $N_l \leqslant (2\gamma)^m$ was established in [5]). Thus the multiplicity of the covering by the cubes from $\pi(\gamma)$ does not exceed $\sum_{l=1}^{L} N_l \leqslant (\gamma+1)^m \, L$. ∎

LEMMA 5.4. *Let $\Gamma \subset \mathbf{R}^m$ be a compact s-dimensional surface of class C^1, $0 \leqslant s \leqslant m-1$, and let $\Gamma_u = \Gamma + Q_u(0) = \bigcup_{x \in \Gamma} Q_u(x)$, $u > 0$. If $Q(x_0)$ is a cube that satisfies all the properties of Lemma 5.2, then*

$$|Q(x_0) \cap \Gamma_u|_m \leqslant C t^s u^{m-s}, \qquad 0 < t, u \leqslant 4\sqrt{m}, \tag{5.18}$$

where C is a constant independent of t, u, and x_0.

Proof. First, by applying Besicovich's covering lemma (a special case of Lemma 4.1 for $\Pi(x) = Q_{r(x)}(x)$ [29]) to a family $\{Q_u(x)\}_{x \in \Gamma}$, we can choose a subfamily $\{Q_u(x_j)\}$ that covers Γ, and the multiplicity of the covering of points of $\mathbf{R}^m$ by the subfamily does not exceed $\theta_5(m)$. It is easy to verify that $\bigcup_j Q_u(x_j) \subseteq \Gamma_u \subseteq \bigcup_j Q_{2u}(x_j)$.

Next, Lemma 5.3 shows that the multiplicity of the covering of points of $\mathbf{R}^m$ by $\pi = \{Q_{2u}(x_j)\}$ does not exceed $\theta_6(m)$. Hence, by [8], $\pi = \bigcup_{i=1}^{N} \pi_i$, where π_i, $1 \leqslant i \leqslant N = N(m)$, are packings.

Then for every $Q_t(x_0)$, $x_0 \in \Gamma$, $0 < t \leqslant \delta(\Gamma)$, where $\delta(\Gamma)$ is the constant from Lemma 5.2, we obtain

$$|Q_t(x_0) \cap \Gamma_u|_m \leqslant \sum_{i=1}^{N} \sum_{Q' \in \pi_i,\, Q_t(x_0) \cap Q' \neq \varnothing} |Q'|_m \leqslant (2u)^m \sum_{i=1}^{N} k_i, \tag{5.19}$$

where $k_i = \dim\{Q' \in \pi_i : Q_t(x_0) \cap Q' \neq \varnothing\}$, $1 \leqslant i \leqslant N$. Further, it follows from Lemma 5.2 that for $0 < u$, $t \leqslant \delta(\Gamma)$,

$$|Q_t(x_0) \cap \Gamma|_s \leqslant Ct^s, \qquad |Q' \cap \Gamma|_s \geqslant (2u)^s. \tag{5.20}$$

Taking into account that each π_i is a family of mutually disjoint cubes, we obtain from (5.20)

$$k_i \leqslant |Q_t(x_0) \cap \Gamma|_s / \min_{Q' \in \pi_i} |Q' \cap \Gamma|_s \leqslant C(t/u)^s, \qquad 1 \leqslant i \leqslant N. \tag{5.21}$$

Therefore, (5.19) and (5.21) yield (5.18) for $0 < u$, $t \leqslant \delta(\Gamma)$. Thus the lemma follows. ▮

Proof of Theorem 5.2. First we prove statement (b). Let Q be a subcube of Q_0 with the edge lenth $t \in (0, 1]$. If $\rho(Q, \Gamma) > t\sqrt{m}$, then for $W(x) = (\rho(x, \Gamma))^\alpha$ and every $M \in (0, 1)$, we have

$$\begin{aligned} W_Q^*(t^m)/W_Q^*(Mt^m) &\leqslant \max_{x \in Q} W(x) / \min_{x \in Q} W(x) \\ &\leqslant ((\rho(Q, \Gamma) + \sqrt{m}\, t)/\rho(Q, \Gamma))^\alpha < 2^\alpha. \end{aligned} \tag{5.22}$$

Let now $0 \leqslant \rho(Q, \Gamma) \leqslant \sqrt{m}\, t$. Then there exists $x_0 \in \Gamma$ such that $Q \subseteq Q_{4\sqrt{m}\, t}(x_0)$. Next, taking account of Lemma 5.4, we obtain

$$\begin{aligned} |\{x \in Q : (\rho(x, \Gamma))^\alpha \leqslant u\}|_m &\leqslant |\{x \in Q_{4\sqrt{m}\, t}(x_0) : \rho(x, \Gamma) \leqslant u^{1/\alpha}\}|_m \\ &\leqslant |Q_{4\sqrt{m}\, t}(x_0) \cap \Gamma_{u^{1/\alpha}}|_m \leqslant Ct^s u^{(m-s)/\alpha}. \end{aligned} \tag{5.23}$$

This implies

$$W_Q^*(\tau) \geqslant C(\tau t^{-s})^{\alpha/(m-s)}, \qquad \tau \in (0, t^m). \tag{5.24}$$

Further taking account of (5.24), we obtain for every $M \in (0, 1)$,

$$W_Q^*(t^m)/W_Q^*(Mt^m) \leqslant C(4\sqrt{m}\, t)^\alpha / (Mt^{m-s})^{\alpha/(m-s)} \leqslant C. \tag{5.25}$$

Therefore statement (b) follows from (5.22), (5.25), and Definition 2.2.

To prove statement (a), we need the following relations

$$\begin{aligned} &|\{x \in Q_0 : (\rho(x, \Gamma))^\alpha (\rho(x, \Gamma) + n^{-2m})^\beta \leqslant u\}|_m \\ &\quad = |\{x \in Q_0 : (\rho(x, \Gamma))^{\alpha/\beta + 1} + n^{-2m}(\rho(x, \Gamma))^{\alpha/\beta} \leqslant u^{1/\beta}\}|_m \\ &\quad \leqslant |\{x \in Q_0 : \rho(x, \Gamma) \leqslant u^{1/(\alpha+\beta)}\}|_m + |\{x \in Q_0 : \rho(x, \Gamma) \leqslant u^{1/\alpha} n^{2m\beta/\alpha}\}|_m. \end{aligned} \tag{5.26}$$

Applying (5.23) to $Q = Q_0$, we obtain from (5.26)

$$
\begin{aligned}
|\{x \in Q_0 : W_{\alpha,\beta,m}(x) \leqslant u\}|_m &\leqslant C(u^{(m-s)/(\alpha+\beta)} + u^{(m-s)/\alpha} n^{2m(m-s)\beta/\alpha}) \\
&\leqslant C \begin{cases} u^{(m-s)/(\alpha+\beta)}, & 0 \leqslant u < n^{-2m(\alpha+\beta)} \\ u^{(m-s)/\alpha} n^{2m(m-s)\beta/\alpha}, & u \geqslant n^{-2m(\alpha+\beta)}. \end{cases}
\end{aligned}
\tag{5.27}
$$

Then (5.27) yields (5.14). ▌

COROLLARY 5.3. (a) *If $F(Q_0)$ is a WRI space, then for a polynomial $P \in \mathscr{P}_{n,m}$ and $\lambda = 2m((\alpha + m - s)/(m-s) + \beta)$,*

$$
\|P\|_{C(Q_0)} \leqslant C\psi_F(n^\lambda)\, \|P\|_{F(Q_0)}. \tag{5.28}
$$

In particular, for $0 < p \leqslant q \leqslant \infty$,

$$
\|P\|_{L_{q,W_{\alpha,\beta,m}}(Q_0)} \leqslant cn^{\lambda(1/p-1/q)}\, \|P\|_{L_{p,W_{\alpha,\beta,m}}(Q_0)}. \tag{5.29}
$$

(b) *For $W = W_{\alpha,0,m}$, a subcube $Q \subseteq Q_0$, a polynomial $P \in \mathscr{P}_{n,m}$, and a set $E \subseteq Q$, $|E|_m > 0$,*

$$
\|P\|_{C(Q)} \leqslant C(\mu V/\mu E)^n\, \|P\|_{C(E)}.
$$

Proof. Note first that statement (b) immediately follows from Theorems 5.2(b) and 2.2(b). Next, (5.14) implies $\int_0^{n^{-2m}} W^*_{\alpha,\beta,m}(\tau)\, d\tau \geqslant Cn^{-\lambda}$. Together with Corollary 3.1(a) and (c), this yields (5.28) and (5.29). ▌

Inequality (5.29) for Q_0, replaced by a convex body V, and $\Gamma = \partial V$, $s = m-1$ (that is $\lambda = 2m(\alpha+1)+\beta$), was established in [23]. For more general sets Ω, the weight $W_{\alpha,0,m}$ and $\Gamma = \partial\Omega$, $s = m-1$, (5.29) was proved in [11, 12]. The one-dimensional analogues of (5.29) for $W_{\alpha,0,1}$ and $W_{0,\beta,1}$, $\Gamma = \{-1, 1\}$, $s = 0$, were obtained in [13, 14].

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REFERENCES

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
2. E. F. Beckenbach and R. Bellman, "Inequalities," Springer-Verlag, Berlin, 1961.
3. P. Borwein and T. Erdelyi, "Polynomials and Polynomial Inequalities," Springer-Verlag, New York, 1995.

4. Yu. A. Brudnyi, On a rearrangement of a smooth function, *Uspekhi Mat. Nauk* **27** (1972), 165–166. [Russian]
5. Yu. A. Brudnyi, Adaptive approximation of functions with singularities, *Trudy Mosk. Matem. Obch.* **55** (1994). [Russian; English translation in *Trans. Moscow Math. Soc.* (1994), 123–186]
6. Yu. A. Brudnyi and M. I. Ganzburg, On an extremal problem for polynomials in n variables, *Izv. Akad. Nauk SSSR* **37** (1973), 344–355. [Russian; English translation in *Math. USSR-Izv* **7** (1973), 345–356]
7. Yu. A. Brudnyi and M. I. Ganzburg, On the exact inequality for polynomials of many variables, *in* "Proceedings of 7th Winter Meeting on Function Theory and Functional Analysis, Drogobych, 1974," pp. 118–123, Moscow, 1976. [Russian]
8. Yu. A. Brudnyi and B. D. Kotlyar, A problem of combinatorial geometry, *Sibirsk. Mat. Zh.* **11** (1970), 1191–1193. [Russian; English translation in *Siberian Math. J.* **11** (1970)]
9. K. M. Chong and N. M. Rice, "Equimeasurable Rearrangements of Functions," Queen's University, Kingston, Ontario, Canada, 1971.
10. E. T. Copson, "Asymptotic Expansions," Cambridge Univ. Press, Cambridge, UK, 1967.
11. I. K. Daugavet, On Markov-Nikolskii-type inequalities for algebraic polynomials in the multidimensional case, *Dokl. Akad. Nauk SSSR* **207** (1972), 521–522. [Russian; English translation in *Soviet Math. Dokl.* **13** (1972), 1548–1550]
12. I. K. Daugavet, Some inequalities for polynomials in the multidimensional case, *Numer. Methods* (*Leningrad Univ.*) **10** (1976), 3–26. [Russian]
13. I. K. Daugavet and C. Z. Rafalson, Some Markov- and Nikolskii-type inequalities for algebraic polynomials, *Vestnik Leningrad. Univ.* **1** (1972), 15–25. [Russian]
14. I. K. Daugavet and C. Z. Rafalson, On some inequalities for algebraic polynomials, *Vestnik Leningrad. Univ.* **19** (1974), 18–24. [Russian]
15. R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, New York, 1993.
16. V. K. Dzyadyk, On a constructive characteristic of functions satisfying the Lipschitz condition α $(0<\alpha<1)$ on a finite segment of the real axis, *Izv. Akad. Nauk SSSR* (*ser Mat.*) **20** (1956), 623–642. [Russian]
17. T. Erdelyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.* **45** (1992), 255–264.
18. T. Erdelyi and P. Nevai, Generalized Jacobi weights, Christoffel functions, and zeros of orthogonal polynomials, *J. Approx. Theory* **69** (1992), 111–132.
19. G. Freud, "Orthogonal Polynomials," Pergamon Press, Oxford, 1971.
20. M. I. Ganzburg, "Local Inequalities for Polynomials and Differential Properties of Functions of Many Variables," Doctoral Thesis, Dniepropetrovsk State Univ., Dniepropetrovsk, 1974. [Russian]
21. M. I. Ganzburg, Some inequalities for polynomials and entire functions of finite degree in symmetric spaces, *in* "The Theory of the Approximation of Functions (Proc. Intern. Conf., Kaluga, 1975)," pp. 104–107, Nauka, Moscow, 1977. [Russian]
22. M. I. Ganzburg, An exact inequality for the increasing rearrangement of a polynomial in m variables, *Teor. Funk., Funk. Anal., Prilozh.* (*Kharkov*) **31** (1978), 16–24. [Russian]
23. M. I. Ganzburg, Inequalities between weighted normes of polynomials of many variables, *in* "Studies in the Theory of Functions of Several Real Variables" (Yu. A. Brudnyi, Ed.), pp. 25–34, Yaroslavl. State Univ., Yaroslavl, 1982. [Russian]
24. M. I. Ganzburg, Polynomial inequalities on measurable sets and their applications, *Constr. Approx.*, to appear.
25. P. Goetgheluck, Polynomial inequalities and Markov's inequality in weighted L^p-spaces, *Acta Math. Acad. Sci. Hungar.* **33** (1979), 325–331.

26. P. Goetgheluck, Une inégalité polynômiale en plusieurs variables, *J. Approx. Theory* **40** (1984), 161–172.
27. P. Goetgheluck, Polynomial inequalities on general subsets of $\mathbf{R}^N$, *Coll. Mat.* **57** (1989), 127–136.
28. P. Goetgheluck, On the problem of sharp exponents in multivariate Nikolskii-type inequalities, *J. Approx. Theory* **77** (1994), 167–178.
29. M. de Guzman, "Differentiation of Integrals in R^n," Springer-Verlag, Berlin, 1975.
30. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, "Interpolation of Linear Operators," Amer. Math. Soc., Providence, RI, 1982.
31. A. Kroo and D. Schmidt, Some extremal problems for multivariate polynomials on convex bodies, *J. Approx. Theory* **90** (1997), 415–434.
32. V. S. Lasher, On Peano derivatives in $L_p(E_n)$, *Studia Math.* **29** (1968), 195–201.
33. G. K. Lebed, Some inequalities for trigonometric and algebraic polynomials and their derivatives, *Trudy MIAN SSSR* **134** (1975), 142–160. [Russian]
34. G. G. Lorentz, M. v. Golitschek, and Y. Makavoz, "Constructive Approximation: Advanced Problems," Springer-Verlag, New York, 1996.
35. D. S. Lubinsky, Ideas of weighted polynomial approximation on $(-\infty, \infty)$, *in* "Approximation Theory VIII, Vol. I: Approximation and Interpolation" (C. K. Chui and L. L. Schumaker, Eds.), pp. 371–396, World Scientific, Singapore, 1995.
36. D. S. Lubinsky and E. B. Saff, Markov–Bernstein and Nikolskii inequalities, and Christoffel functions for exponential weights on $(-1, 1)$, *SIAM J. Math. Anal.* **24** (1993), 528–556.
37. G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, "Topics in Polynomials: Extremal Problems, Inequalities, Zeros," World Scientific, Singapore, 1994.
38. A. P. Morse, Perfect blankets, *Trans. Amer. Math. Soc.* **6** (1947), 418–442.
39. N. S. Nadirashvili, On a generalization of Hadamard's three-circle theorem, *Vestnik Mosk. Univ.* **31**, 3 (1976), 39–42. [Russian; English translation in *Moscow Univ. Math. Bull.* **31**, 3 (1976), 30–32]
40. N. S. Nadirashvili, Estimating solutions of elliptic equations with analytic coefficients bounded on some set, *Vestnik Mosk. Univ.* **34**, 2 (1979), 42–46. [Russian; English translation in *Moscow Univ. Math. Bull.* **34**, 2 (1979), 44–48]
41. F. L. Nazarov, Local estimates of exponential polynomials and their applications to inequalities of uncertainly principle type, *Algebra i Analis* **5**, 4 (1993). [Russian; English translation in *St. Petersburg Math. J.* **5**, 4 (1994), 663–717]
42. E. Remez, Sur une propriété extrémale des polynômes de Tchebychef, *Comm. Inst. Sci. Kharkov* **13**, 4 (1936), 93–95.
43. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
44. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, New York, 1997.
45. I. Schur, Über des maximum des absoluten Betrages eines Polynoms in einen gegebenen Interval, *Math. Z.* **4** (1919), 271–287.
46. R. Sikorski, "Advanced Calculus: Functions of Several Variables," Polish Scientific, Warsaw, 1969.
47. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Pergamon Press, New York, 1963.